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# CALCULATION OF ELECTRIC AND MAGNETIC FIELDS DUE TO AN ELECTROJET CURRENT SYSTEM ABOVE A LAYERED EARTH

by

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Abstract

A theoretical model of an electrojet current system is discussed. The model consists of a horizontal sheet current (electrojet) whose intensity and direction have arbitrary dependencies on the space coordinates and on time, and of field-aligned currents above the electrojet which carry extra charges away from (or to) the ionosphere so that the divergence of the total current density vanishes. The earth is described as a half-space with an infinite plane surface, and is assumed to consist of N uniform horizontal layers. An algorithm is presented for the calculation of the induced fields. Rigorous formulas are derived which directly give the total electromagnetic field on the earth's surface caused by the model current system and induction in the earth.

## 1. *Introduction*

The simplest model used in studies of the electromagnetic field caused by an ionospheric electrojet current (auroral or equatorial) consists of an infinitely long horizontal line current above the flat surface of the earth treated as a half-space. The time dependence, if explicitly taken into account, is generally assumed to be harmonic (*e.g.* ALBERTSON & VAN BAELEN, 1970; HERMANCE & PELTIER, 1970). The use of a flat earth implies applicability to local studies in which areas under investigation have dimensions in the order of hundreds of kilometres.

The basic line current model has been extended to horizontal sheet currents by PELTIER & HERMANCE (1971) and by HIBBS & JONES (1973; 1976a; 1976b).

PIRJOLA (1982; 1985a), however, developed the line current model in a different way when he also included a harmonic space dependence in the direction of the current flow. The corresponding generalization was actually involved also in the treatment by WAIT (1980), but a line current was not discussed explicitly.

The space dependence in the direction of the current, though making an important extension of the original line current model, also implies a serious geophysical disadvantage: because of the non-zero divergence of the current significant electric charge accumulation occurs on the line. In reality, the accumulation is prevented by additional currents, the most important of which are parallel to the geomagnetic field, *i.e.* field-aligned currents. In the auroral zones the geomagnetic field is almost vertical. Therefore LEHTO (1983; 1984) improved Pirjola's auroral electrojet model by adding vertical currents starting upwards from the electrojet and making the divergence of the total current vanish. In fact, he did not restrict the treatment to a line current electrojet, but dealt with a horizontal sheet current having arbitrary time and space dependencies. However, there was one limitation which, as Lehto mentioned, is a drawback from the geophysical point of view: the electrojet was assumed to have the same cross-sectional distribution at every moment of time and everywhere along the jet. Numerical results of the electromagnetic field on the earth's surface caused by special cases of Lehto's current model and induction in the earth have been presented by LEHTO (1983) and by PIRJOLA (1985b); LEHTO (1984) contains theory only.

In this paper we will show that Lehto's model can be further developed by allowing the horizontal sheet current electrojet have any time and space dependencies, so the above-mentioned limitation concerning the cross-sectional distribution is released. Also the direction of the electrojet current may vary in time and space. In addition, the field-aligned currents starting upwards from the electrojet may have any (straight) direction, *i.e.* the inclination and declination of the geomagnetic field are arbitrary.

The electromagnetic properties of the earth, in which the electromagnetic induction occurs, vary both vertically and horizontally in practice. The former variation is more important and easier to be taken into account. Therefore the latter is often neglected in theoretical treatments. PIRJOLA (1982) used an earth model in which the properties change arbitrarily in the vertical and »almost arbitrarily» in one horizontal direction. However, he could give only formal solutions to the problem which are not directly usable in numerical computations. In this paper we will study the theoretical problem considering also applicability to numerical calculations. Hence we disregard horizontal variations and assume the earth, which is a half-space with a flat surface, to be composed of N homogeneous horizontal layers. The conductivity, permeability and permittivity of each layer may have any values. An algorithm necessary in numerical computations will be presented for the treatment of the N layers. Thus also concerning the earth model, this paper presents an extension as compared to LEHTO (1983; 1984) who discussed a two-layered earth.

It should be noted that field-aligned currents have generally been included in static models (*e.g.* KISABETH, 1972; KISABETH & ROSTOKER, 1977). In these references time dependence of the currents of the electrojet system was actually also taken into account, but only implicitly. It was done by assuming that the earth, which was treated as a sphere implying a global discussion, contained a perfectly conducting shell at a certain depth. Owing to the infinite conductivity, all explicit time dependence of the induced currents vanished.

The final results of this paper are formulas from which expressions for the total electromagnetic field on the earth's surface, *i.e.* the primary field caused by the electrojet current system plus the field due to induction in the earth, are obtained in a straightforward manner. The calculations are based on rigorous Maxwell's equations including also displacement currents. Owing to the low frequencies associated with geomagnetic phenomena the latter could, however, obviously be neglected, as is usually done except for the above-mentioned papers by Wait, Pirjola and Lehto. But displacement currents can also easily be retained, and the omission might limit the possible applicability of the results to other electromagnetic problems.

This paper is purely theoretical. Numerical calculations will be presented in subsequent papers, and the model is planned to be applied in connection with magnetic data obtained by the so-called EISCAT magnetometer cross in northern Scandinavia (SUCKSDORFF *et al.*, 1984).

## 2. *Description of the current system*

Let us use the coordinate system conventional in geomagnetic studies, *i.e.* the $x$-axis points to the north, the $y$-axis to the east and the $z$-axis downwards into the earth. The earth's surface is an infinite plane where $z = 0$. The horizontal electrojet is an infinitely thin sheet current at height $h$ ($h > 0$, $z = -h$) whose intensity and direction have arbitrary dependencies on $x$- and $y$-coordinates and also on time. Mathematically this is described as

$$\mathbf{j}_{\mathrm{H}}(\mathbf{r},t) = \mathbf{J}(x,y,t)\,\delta(z+h)\,, \tag{1}$$

where

$$\mathbf{J}(x,y,t) = J_x(x,y,t)\,\hat{\mathbf{e}}_x + J_y(x,y,t)\,\hat{\mathbf{e}}_y\,, \tag{2}$$

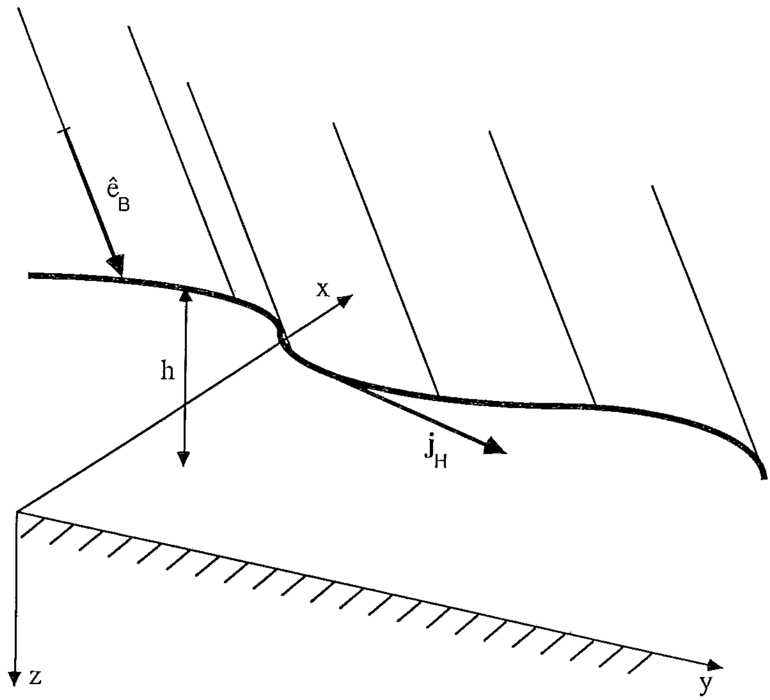


Fig. 1. The current system model in which a horizontal electrojet at height h flows above a half-space earth. Field-aligned currents flow in the constant direction defined by the vector $\hat{e}_B$.

$\delta$ is the Dirac delta distribution, and $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in the $x$- and $y$-directions, respectively. Due to the $x$- and $y$-dependencies of $\mathbf{j}_H$ charge is accumulated into the ionosphere ($\partial\rho/\partial t = -\nabla\cdot\mathbf{j}_H \neq 0$). To avoid this let us add field-aligned currents to the system that carry this extra charge away from (or to) the ionosphere. Thus the total current density must satisfy the equation $\nabla\cdot\mathbf{j}_{tot} = 0$. It is easy to verify that the following current density meets this requirement:

$$\mathbf{j}_{tot}(\mathbf{r},t) = \mathbf{j}_H(\mathbf{r},t) + \frac{1}{\hat{e}_z\cdot\hat{e}_B}\left[\nabla\cdot\mathbf{J}(x-(z+h)\alpha_x, y-(z+h)\alpha_y, t)\right][1-\theta(z+h)]\hat{e}_B, \quad (3)$$

where $\hat{e}_B$ is the direction vector of the field-aligned currents (fig. 1):

$$\hat{e}_B = (\cos D\cos I, \sin D\cos I, \sin I), \quad (4)$$

$D$ and $I$ are the geomagnetic declination and inclination, and

$$\boldsymbol{\alpha} \equiv (\alpha_x, \alpha_y, 0) = \left( \frac{\hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_B}{\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B} , \frac{\hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_B}{\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B} , 0 \right), \tag{5}$$

and $\theta$ is the Heaviside step function.

For later use let us calculate the Fourier transform of $\mathbf{j}_{tot}(\mathbf{r}, t)$. The following conventions for the Fourier transforms with respect to a space coordinate and time are used:

$$f(q,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{iqx} f(x,t) , \tag{6}$$

$$f(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{-i\omega t} f(x,t) . \tag{7}$$

The Fourier transform of $\mathbf{j}_{tot}(\mathbf{r}, t)$ with respect to $x, y$ and $t$ is then:

$$\begin{aligned} \mathbf{j}_{tot}(q_1,q_2,z,\omega) &= \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \; e^{-i\omega t} e^{iq_1 x} e^{iq_2 y} \, \mathbf{j}_{tot}(\mathbf{r},t) \\ &= \mathbf{J}(\mathbf{q},\omega) \, \delta(z+h) - \frac{[1-\theta(z+h)]}{\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B} \, i\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\omega) \, e^{i(z+h)\mathbf{q}\cdot\boldsymbol{\alpha}} \, \hat{\mathbf{e}}_B , \end{aligned} \tag{8}$$

where $\mathbf{J}(\mathbf{q}, \omega)$ is the Fourier transform of $\mathbf{J}(x, y, t)$:

$$\mathbf{J}(\mathbf{q},\omega) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \; e^{-i\omega t} e^{iq_1 x} e^{iq_2 y} \; \mathbf{J}(x,y,t) , \tag{9}$$

and $\mathbf{q} = (q_1, q_2, 0)$.

## 3. *Calculation of the primary fields*

The expression for the magnetic vector potential $\mathbf{A}$ in a medium with no electrical conductivity has been derived, for example, by PANOFSKY & PHILLIPS (1964, pp. 242–244). Applying the same method to a non-zero conductivity $\sigma$, we get

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi} \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega \, \mathrm{d}^3\mathbf{r}' \; \frac{\mathbf{j}(\mathbf{r}',\omega) \, e^{i(\omega t - kR)}}{R} \tag{10}$$

where $\mathbf{j}(\mathbf{r}', \omega)$ is the Fourier transform of the primary electric current density with respect to time, $R$ is the magnitude of the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and the complex wave number $k$ is defined by

$$\begin{cases} k^2 = \omega^2\mu\varepsilon - i\omega\mu\sigma, \\ \mathrm{Im}\, k \leq 0. \end{cases} \tag{11}$$

The condition $\mathrm{Im}\, k \leqslant 0$ is required for the convergence of the integral in eq. (10). Here the Fourier transform $\mathbf{j}(\mathbf{r}', \omega)$ may be further written:

$$\mathbf{j}(\mathbf{r}',\omega) = \frac{1}{2\pi} \int dq_1\, dq_2\; e^{-iq_1x'}\, e^{-iq_2y'}\; \mathbf{j}(q_1,q_2,z',\omega)\,, \tag{12}$$

Substituting eq. (12) into eq. (10) we find

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi}\frac{1}{(2\pi)^{3/2}} \int d\omega\, d^3\mathbf{r}'\, \frac{e^{i(\omega t-kR)}}{R} \int dq_1\, dq_2\; e^{-iq_1x'}\, e^{-iq_2y'}\; \mathbf{j}(q_1,q_2,z',\omega)\,, \tag{13}$$

where $R = [(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}$. Changing the order of integration and making simple changes of variables $(x' \rightarrow x'+x,\ y' \rightarrow y'+y)$ we get

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int d\omega\, dq_1\, dq_2\, e^{i\omega t}\, e^{-iq_1x}\, e^{-iq_2y}\;\; \mathbf{A}(\mathbf{q},z,\omega)\,, \tag{14}$$

where

$$\mathbf{A}(\mathbf{q},z,\omega) = \frac{\mu}{4\pi} \int d^3\mathbf{r}'\, \frac{e^{-ikR}}{R}\, e^{-iq_1x'}\, e^{-iq_2y'}\; \mathbf{j}(q_1,q_2,z',\omega)\,, \tag{15}$$

and now $R = [x'^2+y'^2+(z-z')^2]^{1/2}$. The integrals over $x'$ and $y'$ may be immediately performed by using polar coordinates and integrating first over the polar angle then over the radial distance (WATSON, 1948, p. 416):

$$\begin{aligned} \mathbf{A}(\mathbf{q},z,\omega) &= \frac{\mu}{4\pi} \int dz'\, \mathbf{j}(q_1,q_2,z',\omega) \int dx'\, dy'\, \frac{e^{-ikR}}{R}\, e^{-iq_1x'}\, e^{-iq_2y'} \\ &= \frac{\mu}{4\pi} \int dz'\, \mathbf{j}(q_1,q_2,z',\omega)\cdot 2\pi\, \frac{e^{-|z-z'|\sqrt{q^2-k^2}}}{\sqrt{q^2-k^2}}\,, \end{aligned} \tag{16}$$

where $q^2 = q_1^2 + q_2^2$ and Re $\sqrt{q^2 - k^2} > 0$ (again for convergence).

Defining $\xi_0 \equiv \sqrt{q^2 - k^2}$ and substituting the expression of $\mathbf{j}_{tot}(q_1, q_2, z', \omega)$ (eq. (8)) into eq. (16) we get:

$$
\begin{aligned}
\mathbf{A}(\mathbf{q},z,\omega) &= \\
&= \frac{\mu}{2\xi_o} \int_{-\infty}^{\infty} dz'\, e^{-|z-z'|\xi_o} \left[ \mathbf{J}(\mathbf{q},\omega)\, \delta(z'+h) - \frac{[1-\theta(z'+h)]}{\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B}\, i\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\omega)\, e^{i(z'+h)\mathbf{q}\cdot\boldsymbol{\alpha}}\, \hat{\mathbf{e}}_B \right] \\
&= \frac{\mu}{2\xi_o} \left[ e^{-|z+h|\xi_o}\, \mathbf{J}(\mathbf{q},\omega) - \frac{i\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\omega)}{\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B}\, \hat{\mathbf{e}}_B \int_{-\infty}^{-h} dz'\, e^{-|z-z'|\xi_o} e^{i(z'+h)\mathbf{q}\cdot\boldsymbol{\alpha}} \right] \qquad (17) \\
&= \frac{\mu}{2\xi_o}\, e^{-(z+h)\xi_o} \left[ \mathbf{J}(\mathbf{q},\omega) - \frac{i\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\omega)}{(i\mathbf{q}\cdot\boldsymbol{\alpha}+\xi_o)(\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_B)}\, \hat{\mathbf{e}}_B \right], \qquad (z > -h).
\end{aligned}
$$

This is the desired expression for the vector potential caused by the primary current density (eq. (3)). It is easy to verify that the field $\mathbf{A}(\mathbf{r}, t)$ given by eqs. (14) and (17) satisfies the gauge condition $\nabla \cdot \mathbf{A} = 0$. The scalar potential $\phi$ may be chosen to be identically zero since $\rho \equiv 0$. Thus the expressions for the electric and magnetic fields are easily derived from the formulae (PANOFSKY & PHILLIPS, 1964, p. 240):

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}, \\
\mathbf{B}(\mathbf{r},t) &= \nabla \times \mathbf{A}(\mathbf{r},t)\,.
\end{aligned} \qquad (18)
$$

The corresponding formulae for the Fourier transformed primary fields are then:

$$
\begin{aligned}
\mathbf{E}(\mathbf{q},z,\omega) &= -i\omega \mathbf{A}(\mathbf{q},z,\omega)\,, \\
\mathbf{B}(\mathbf{q},z,\omega) &= -i\mathbf{Q} \times \mathbf{A}(\mathbf{q},z,\omega)\,,
\end{aligned} \qquad (19)
$$

where $\mathbf{Q} \equiv (q_1, q_2, -i\xi_0)$.

## 4. *Calculation of the induced fields*

Let us describe the earth as a half-space with an infinite plane surface, and assume that it consists of N uniform horizontal layers with electromagnetic parameters $\epsilon_i, \mu_i, \sigma_i$ and thicknesses $h_i$ (fig. 2). Due to a possible time dependence of the primary current density $\mathbf{j}_{tot}$ currents and charges are induced into the earth, which in turn generate magnetic and electric fields. We denote the total (*i.e.* primary plus induced) electric and magnetic fields in the $i^{\text{th}}$ layer of the earth by $\mathbf{E}^i(\mathbf{r}, t)$ and $\mathbf{B}^i(\mathbf{r}, t)$, $i = 1,\dots,N$, and the notations $\mathbf{E}^0(\mathbf{r}, t)$ and $\mathbf{B}^0(\mathbf{r}, t)$ refer to the induced fields above the earth.

The following wave equations are satisfied in each layer $i = 0,\dots, N$:

$$\nabla^2\mathbf{E}^i - \mu_i\varepsilon_i \frac{\partial^2\mathbf{E}^i}{\partial t^2} - \mu_i\sigma_i \frac{\partial\mathbf{E}^i}{\partial t} = 0 \ , \tag{20}$$

$$\nabla^2\mathbf{B}^i - \mu_i\varepsilon_i \frac{\partial^2\mathbf{B}^i}{\partial t^2} - \mu_i\sigma_i \frac{\partial\mathbf{B}^i}{\partial t} = 0 \ . \tag{21}$$

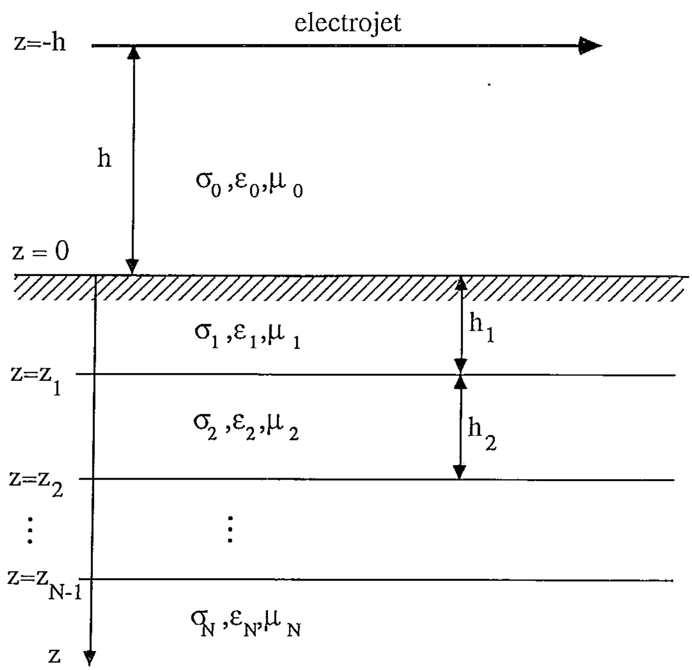


Fig. 2. Model of the earth composed of N horizontal layers.

Writing

$$\mathbf{E}^i(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int d\omega\, dq_1\, dq_2\, e^{i\omega t} e^{-iq_1 x} e^{-iq_2 y}\ \mathbf{E}^i(\mathbf{q},z,\omega)\,, \tag{22}$$

$$\mathbf{B}^i(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int d\omega\, dq_1\, dq_2\, e^{i\omega t} e^{-iq_1 x} e^{-iq_2 y}\ \mathbf{B}^i(\mathbf{q},z,\omega)\,, \tag{23}$$

we see that the Fourier transformed fields $\mathbf{E}^i(\mathbf{q},z,\omega)$ and $\mathbf{B}^i(\mathbf{q},z,\omega)$ satisfy the equations

$$\frac{\partial^2 \mathbf{E}^i(\mathbf{q},z,\omega)}{\partial z^2} - (q^2 - k_i^2)\,\mathbf{E}^i(\mathbf{q},z,\omega) = 0\,, \tag{24}$$

$$\frac{\partial^2 \mathbf{B}^i(\mathbf{q},z,\omega)}{\partial z^2} - (q^2 - k_i^2)\,\mathbf{B}^i(\mathbf{q},z,\omega) = 0\,, \tag{25}$$

where $q^2 = q_1^2 + q_2^2$ and

$$k_i^2 = \omega^2 \mu_i \varepsilon_i - i\omega\mu_i\sigma_i\,. \tag{26}$$

Let us consider only the $z$-components of $\mathbf{E}^i(\mathbf{q}, z, \omega)$ and $\mathbf{B}^i(\mathbf{q}, z, \omega)$. The general solutions for them may be written:

$$E_z^i(\mathbf{q},z,\omega) = D_i(\mathbf{q},\omega)\, e^{\xi_i(z-z_{i-1})} + G_i(\mathbf{q},\omega)\, e^{-\xi_i(z-z_{i-1})}\,, \tag{27}$$

$$B_z^i(\mathbf{q},z,\omega) = Q_i(\mathbf{q},\omega)\, e^{\xi_i(z-z_{i-1})} + R_i(\mathbf{q},\omega)\, e^{-\xi_i(z-z_{i-1})}\,, \tag{28}$$

where

$$\begin{cases} \xi_i = \sqrt{q^2 - k_i^2}\,, \\ \mathrm{Re}\,\xi_i \geq 0\,, \end{cases} \tag{29}$$

and $z_{-1} \equiv 0$. The expressions for the $x$- and $y$-components of the Fourier transformed fields may be found from the Maxwell's equations:

$$\begin{cases} -iq_1 E_x - iq_2 E_y = -\dfrac{\partial E_z}{\partial z}\,, \\[2mm] -iq_1 B_x - iq_2 B_y = -\dfrac{\partial B_z}{\partial z}\,, \\[2mm] -iq_1 B_y + iq_2 B_x = \mu(\sigma + i\omega\varepsilon) E_z\,, \\[2mm] -iq_1 E_y + iq_2 E_x = -i\omega B_z\,. \end{cases} \tag{30}$$

These equations are valid for both the induced and primary fields (when $z > -h$).

Equations (27) and (28) involve 4(N+1) unknown functions $D_i, G_i, Q_i, R_i$; $i=0,\dots,N$. The boundary conditions on the electric and magnetic fields determine these uniquely:

1° The total field must vanish when $z\to\infty$, and the same holds good of the induced field in the air when $z \to -\infty$. This implies

$$\begin{cases} G_0(q,\omega) \equiv 0 , \\ R_0(q,\omega) \equiv 0 , \end{cases} \qquad \begin{cases} D_N(q,\omega) \equiv 0 , \\ Q_N(q,\omega) \equiv 0 . \end{cases} \tag{31}$$

2° The components $E_x(\mathbf{r},t)$, $E_y(\mathbf{r},t)$, $B_x(\mathbf{r},t)/\mu$ and $B_y(\mathbf{r},t)/\mu$ of the total fields must be continuous at $z=z_i$, $i=0,\dots,N-1$. Therefore also the corresponding Fourier transformed fields must be continuous which according to eq. (30) implies the continuity of $\partial E_z(q,z,\omega)/\partial z$, $(\sigma+i\omega\varepsilon)E_z(q,z,\omega)$, $\mu^{-1}\partial B_z(q,z,\omega)/\partial z$ and $B_z(q,z,\omega)$. This gives us the rest 4N conditions.

The total fields in the air are $\mathbf{E}+\mathbf{E}^0$ and $\mathbf{B}+\mathbf{B}^0$ where $\mathbf{E}$ and $\mathbf{B}$ are the primary electric and magnetic fields, and in the $i^{\text{th}}$ layer $\mathbf{E}^i$ and $\mathbf{B}^i$. Therefore the boundary conditions give:

$$\xi_i(D_i e^{\xi_i h_i} - G_i e^{-\xi_i h_i}) + \delta_{i,0}\left.\frac{\partial E_z(q,z,\omega)}{\partial z}\right|_{z=0} = \xi_{i+1}(D_{i+1}-G_{i+1}) , \tag{32}$$

$$\begin{aligned}\mu_i(\sigma_i+i\omega\varepsilon_i)(D_i e^{\xi_i h_i} + G_i e^{-\xi_i h_i}) + \delta_{i,0}\,\mu_0(\sigma_0+i\omega\varepsilon_0)\,E_z(q,z{=}0,\omega) &= \\ = \mu_{i+1}(\sigma_{i+1}+i\omega\varepsilon_{i+1})(D_{i+1}+G_{i+1}),&\end{aligned} \tag{33}$$

$$\frac{\xi_i}{\mu_i}(Q_i e^{\xi_i h_i} - R_i e^{-\xi_i h_i}) + \delta_{i,0}\frac{1}{\mu_0}\left.\frac{\partial B_z(q,z,\omega)}{\partial z}\right|_{z=0} = \frac{\xi_{i+1}}{\mu_{i+1}}(Q_{i+1}-R_{i+1}) , \tag{34}$$

$$Q_i e^{\xi_i h_i} + R_i e^{-\xi_i h_i} + \delta_{i,0}\,B_z(q,z{=}0,\omega) = Q_{i+1}+R_{i+1} . \tag{35}$$

Here $\delta_{i,0}$ is the Kronecker delta symbol. From these equations it is easy to solve $D_{i+1}$ and $G_{i+1}$ as functions of $D_i$ and $G_i$, and $Q_{i+1}$ and $R_{i+1}$ as functions of $Q_i$ and $R_i$. The results are conveniently written in matrix form:

$$\begin{pmatrix} D_{i+1} \\ G_{i+1} \end{pmatrix} = \begin{pmatrix} \gamma_i^+ e^{\xi_i h_i} & \gamma_i^- e^{-\xi_i h_i} \\ \gamma_i^- e^{\xi_i h_i} & \gamma_i^+ e^{-\xi_i h_i} \end{pmatrix} \begin{pmatrix} D_i \\ G_i \end{pmatrix} + \delta_{i,0}\,E_z(q,z{=}0,\omega)\begin{pmatrix} \gamma_0^- \\ \gamma_0^+ \end{pmatrix} , \tag{36}$$

$$\begin{pmatrix} Q_{i+1} \\ R_{i+1} \end{pmatrix} = \begin{pmatrix} \beta_i^+ e^{\xi_i h_i} & \beta_i^- e^{-\xi_i h_i} \\ \beta_i^- e^{\xi_i h_i} & \beta_i^+ e^{-\xi_i h_i} \end{pmatrix} \begin{pmatrix} Q_i \\ R_i \end{pmatrix} + \delta_{i,0} B_z(\mathrm{q},z{=}0,\omega) \begin{pmatrix} \beta_0^- \\ \beta_0^+ \end{pmatrix} , \tag{37}$$

where we already used eqs. (17) and (19) for the evaluation of the derivatives of the Fourier transformed primary fields, and defined

$$\gamma_i^{\pm} \equiv \frac{1}{2} \left[ \frac{\mu_i(\sigma_i + i\omega\varepsilon_i)}{\mu_{i+1}(\sigma_{i+1} + i\omega\varepsilon_{i+1})} \pm \frac{\xi_i}{\xi_{i+1}} \right] , \tag{38}$$

$$\beta_i^{\pm} \equiv \frac{1}{2} \left[ 1 \pm \frac{\mu_{i+1}}{\mu_i} \frac{\xi_i}{\xi_{i+1}} \right] , \tag{39}$$

In order to evaluate the electromagnetic field at the surface of the earth we need to know the coefficients $D_0(\mathrm{q}, \omega)$ and $Q_0(\mathrm{q}, \omega)$. Thus we have to study equations (36) and (37). Since they are similar in nature let us consider only eq. (36).

Let $\mathsf{A}_i$ denote the complex 2x2 matrix multiplying the vector $(D_i\, G_i)$ in eq. (36). Remembering that $G_0 = D_\mathrm{N} = 0$ eq. (36) implies

$$\begin{aligned} \begin{pmatrix} 0 \\ G_\mathrm{N} \end{pmatrix} &= \mathsf{A}_{\mathrm{N}-1} \begin{pmatrix} D_{\mathrm{N}-1} \\ G_{\mathrm{N}-1} \end{pmatrix} = \mathsf{A}_{\mathrm{N}-1} \mathsf{A}_{\mathrm{N}-2} \begin{pmatrix} D_{\mathrm{N}-2} \\ G_{\mathrm{N}-2} \end{pmatrix} = \ldots = \\ &= \mathsf{A}_{\mathrm{N}-1} \ldots \mathsf{A}_1 \left[ \mathsf{A}_0 \begin{pmatrix} D_0 \\ 0 \end{pmatrix} + E_z(\mathrm{q},z{=}0,\omega) \begin{pmatrix} \gamma_0^- \\ \gamma_0^+ \end{pmatrix} \right] . \end{aligned} \tag{40}$$

The upper component of this vector equation involves only $D_0$. Thus it can readily be solved:

$$D_0(\mathrm{q},\omega) = - \frac{(\mathsf{A}_{\mathrm{N}-1} \ldots \mathsf{A}_1)_{11}\, \gamma_0^- + (\mathsf{A}_{\mathrm{N}-1} \ldots \mathsf{A}_1)_{12}\, \gamma_0^+}{(\mathsf{A}_{\mathrm{N}-1} \ldots \mathsf{A}_1)_{11}\, \gamma_0^+ + (\mathsf{A}_{\mathrm{N}-1} \ldots \mathsf{A}_1)_{12}\, \gamma_0^-} E_z(\mathrm{q},z{=}0,\omega) . \tag{41}$$

Similarly from equation (37) we get

$$Q_0(\mathrm{q},\omega) = - \frac{(\mathsf{C}_{\mathrm{N}-1} \ldots \mathsf{C}_1)_{11}\, \beta_0^- + (\mathsf{C}_{\mathrm{N}-1} \ldots \mathsf{C}_1)_{12}\, \beta_0^+}{(\mathsf{C}_{\mathrm{N}-1} \ldots \mathsf{C}_1)_{11}\, \beta_0^+ + (\mathsf{C}_{\mathrm{N}-1} \ldots \mathsf{C}_1)_{12}\, \beta_0^-} B_z(\mathrm{q},z{=}0,\omega) , \tag{42}$$

where $C_i$ is the matrix multiplying the vector $(Q_i\ R_i)$ in eq. (37). If N = 1 then the matrix products $A_{N-1} \dots A_1$ and $C_{N-1} \dots C_1$ in eqs. (41) and (42) must be replaced by a 2x2 unit matrix.

As can be seen from equations (27) and (28) the coefficients $D_0$ and $Q_0$ are in fact the Fourier transforms of the induced fields $E_z^0(\mathbf{r}, t)$ and $B_z^0(\mathbf{r}, t)$ on the earth's surface. Eqs. (41) and (42) thus express the fact that the induced fields are linearly dependent on the primary fields, and the coefficients are fairly simple expressions of the electromagnetic parameters of the various layers. Furthermore, the coefficients as given in eqs. (41) and (42) are easily applicable in numerical computations.

In summary, the algorithm for computing the total electric and magnetic fields on the earth's surface is as follows: First one must define the horizontal current density eq. (2) and compute its Fourier transform eq. (9). Expressions for the primary electric and magnetic fields $\mathbf{E}(\mathbf{q}, z=0, \omega)$ and $\mathbf{B}(\mathbf{q}, z=0, \omega)$ are then derived from equations (17) and (19). The z-components of the induced fields $E_z^0(\mathbf{q}, z=0, \omega)$ and $B_z^0(\mathbf{q}, z=0, \omega)$ are computed from eqs. (41) and (42), after which the $x$- and $y$-components are solved from eq. (30), where for the induced fields $\partial E_z^0/\partial z = \xi_0 E_z^0$ and $\partial B_z^0/\partial z = \xi_0 B_z^0$. Finally one must take the Fourier transforms with respect to $q_1, q_2$ and $\omega$ to get the total electric and magnetic fields at the specified point $(x, y)$ and at time $t$.

The applicability of the present algorithm for calculating the electromagnetic field at the earth's surface requires that one is able to calculate the Fourier transform of the horizontal current density eq. (9). For simple geometrical situations this is usually easy. For example, if we have an infinitely long straight constant line current pointing in the $y$-direction, which, in addition, is moving in the positive $x$-direction with velocity $v$, then $J_x(x, y, t) = 0$ and $J_y(x, y, t) = I\,\delta(x-vt)$. The Fourier transformed horizontal current in this simple case is then $\mathbf{J}(\mathbf{q}, \omega) = \sqrt{2\pi}\,\delta(q_2)\cdot\delta(\omega - q_1 v)\,\hat{e}_y$. The expressions for the electric and magnetic fields thus reduce to single integrals over the variable $q_1$.

## 5. *Conclusions*

This paper deals theoretically with a model of an electrojet current system. The earth is treated as an N-layered half-space with a flat surface implying applicability to local studies. The electrojet system situated at any height above the earth's surface consists of a horizontal sheet current (*i.e.* the electrojet), whose intensity and direction have arbitrary time and space dependencies, and of straight field-aligned currents preventing the accumulation of electric charge in the ionosphere and having any direction (*i.e.* any inclination and declination). The direction is the same at every point and time. The final results of this paper are rigorous ·

formulas which directly give the total (*i.e.* primary plus earth-induced) electromagnetic field on the earth's surface.

As compared to previously published works dealing with electrojet models of similar type this treatment has the following advantages:

1. The behaviour of the electrojet is arbitrary with respect to both time and space, *e.g.* the shape of the electrojet need not be straight.
2. The field-aligned currents recognize that the geomagnetic field is not exactly vertical even in the auroral zones. In fact the present model might be used in equatorial studies, too.
3. Induction in a layered earth having any number of layers is explicitly taken into account.

In future the model presented in this paper could further be developed as follows:

1. Lateral inhomogeneities in the structure of the earth should be included. This will be of particular importance when the model will be used for the EISCAT magnetometer data collected in northern Scandinavia which is situated at the Arctic Ocean.
2. Effects of the curvature of the earth should be discussed.
3. Cases in which the direction of the field-aligned currents changes in time and space could be considered.
4. Cases in which the electrojet current does not flow horizontally, *i.e.* also contains a vertical component, should be studied. It is probable that such a model can be constructed as a superposition of current systems of the type discussed in this paper.

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