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ELECTROMAGNETIC FIELD CAUSED BY A THREE-DIMENSIONAL TIME- AND SPACE-DEPENDENT ELECTROJET CURRENT SYSTEM

by

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Abstract

A time- and space-dependent theoretical model of the high-latitude ionospheric electrojet current system is presented. The earth is described as a half-space with an infinite plane surface, and is assumed to consist of two uniform horizontal layers. The electrojet is taken to be an infinitely thin sheet current whose intensity has an arbitrary dependence on the space coordinate in the direction of the flow and on time. The current density has an arbitrary but constant distribution in the direction perpendicular to the flow. Vertical currents are added above the horizontal jet so that the divergence of the total current density vanishes and accumulation of charge in the ionosphere is avoided. The electromagnetic field caused by the model electrojet current system is calculated at the earth's surface. The secondary field, caused by geomagnetic induction in the earth, is taken into account.

1. *Introduction*

The ionospheric electrojet current system at auroral latitudes consists of a horizontal main current in the E-layer, where the Pedersen and Hall conductivities have their maximum values, and field-aligned currents above the E-layer. These varying currents produce electromagnetic fluctuations both directly and through secondary currents induced in the conducting earth.

A simple basic model is an infinitely long horizontal straight line current oscillating harmonically in time. The resulting electromagnetic field has been studied by LAW & FANNIN (1961, in connection with magnetic micropulsations), HERMANCE &

PELTIER (1970), and PIRJOLA (1982, Chapter 3), among others. In the first two of these publications, the displacement currents are neglected owing to the low frequences occurring in geophysical variations, so the applicability of their results to theoretical and to different practical problems is limited.

The line current results of Hermance and Peltier were applied to an infinite horizontal sheet-current model by PELTIER & HERMANCE (1971). The current density in the sheet was assumed to have a Gaussian (*i.e.* normal) distribution in the direction perpendicular to the current flow, and to be constant in the direction of the flow. HIBBS & JONES (1976) used the formulas of Peltier and Hermance and numerically calculated the fields of different current density distributions by superposing spatially shifted fields of elemental Gaussian sheets.

PIRJOLA (1982, Chapter 4) extended the basic harmonic line-current model in another way by retaining the line form and assuming a harmonic space dependence in the direction of the flow. He did not ignore the displacement currents.

WAIT (1980) used a sheet-current source with a harmonic space dependence in two perpendicular directions. But since he was interested only in developing general expressions for the surface impedance matrix, he did not consider the electrojet at all and did not calculate the total electromagnetic field at the earth's surface.

In all the models described above, as well as in the model presented in this paper, the earth is described as a half-space with an infinite plane surface and a horizontally layered electromagnetic structure, and the horizontal electrojet currents are confined to an infinitely thin sheet at a fixed height near 110 km.

One feature of the above space-dependent models is that the divergence of the current density does not vanish, which implies that charge is accumulated in the ionosphere according to the equation of continuity. In reality, the charge accumulation is very small owing to the field-aligned currents. As PIRJOLA (1982, p. 115) has concluded, the addition of vertical currents (the magnetic field lines are almost radial at high latitudes near the earth's surface) would be a useful extension of the time-dependent models made hitherto. In static models, however, field-aligned currents have often been taken into account (*e.g.* BOSTRÖM (1964), BONNEVIER *et al.* (1970), KISABETH (1972), KAMIDE *et al.* (1981)).

The present paper describes quite a general time- and space-dependent electrojet model. The current intensity in the horizontal sheet has an arbitrary dependence on time and on the space coordinate in the direction of the current flow. The current density has an arbitrary but constant distribution in the direction perpendicular to the flow. Vertical currents are added above the horizontal jet so that the divergence of the total current density vanishes.

To simplify numerical calculations (not presented here), the half-space earth is assumed to be composed of two layers. The permittivity and permeability of the air in the ionosphere and below it are assumed to have vacuum values ϵ_0 and μ_0. The air is also given a slight conductivity. In reality, the conductivity of the air is in the order of $10^{-14}\Omega^{-1}\mathrm{m}^{-1}$ near the earth's surface (ISRAËL, 1971, pp. 95 and 248).

2. *Description of the model*

We will take the earth's surface to be an infinite plane where $z = 0$. The half-space where $z > 0$ comprises the earth itself. The horizontal electrojet is an infinitely thin sheet current at a height h ($h > 0$) with current density

$$\bar{j}_H(\bar{r}, t) = J(y, t) f(x) \delta(z + h) \bar{e}_y \tag{1}$$

where δ is the Dirac delta distribution and f is a density function:

$$f(x) \geqslant 0 \qquad \forall x \in R \tag{2}$$

$$\int_{-\infty}^{\infty} f(x) dx = 1 \tag{3}$$

The quantity $J(y_0, t_0)$ expresses the total current intensity across the line $\{(x, y, z)| y = y_0, z = -h\}$ at time t_0, and $\bar{e}_y$ denotes a unit vector in the y-direction, which is assumed to be the direction of the current flow. The x-direction is then chosen so as to produce an orthogonal right-handed xyz-coordinate system.

To avoid accumulation of charge in the ionosphere, we add vertical currents to the model so as to get the divergence of the total current density to vanish. These currents flow in the region where $z < -h$. The total current density is then

$$\bar{j}(\bar{r}, t) = J(y, t) f(x) \delta(z + h) \bar{e}_y + \frac{\partial J(y, t)}{\partial y} f(x) [1 - \theta(z + h)] \bar{e}_z \tag{4}$$

where θ is the Heavyside step function.

The current system described above contains electric currents up to infinity. There is no problem in the x-direction because, according to equations (2) and (3), $f(x)$ goes to zero as $|x|$ approaches infinity. In the y-direction we can assume that the currents are confined to the region where $-y_0 \leqslant y \leqslant y_0$ ($y_0 > 0$), and that the electrojet current closes itself above the horizontal part along the surface of a half-cylinder with a radius y_0 (Figure 1). The vertical currents are assumed to flow only in the space between the jet and the cylinder, and the whole system can be taken to lie within a volume $V(y_0)$ bordered by a closed surface.

When y_0 is large enough and both $J(y, t)$ and $\partial J(y, t)/\partial y$ are bounded functions, the currents flowing at the cylindrical surface have no observable effect on the

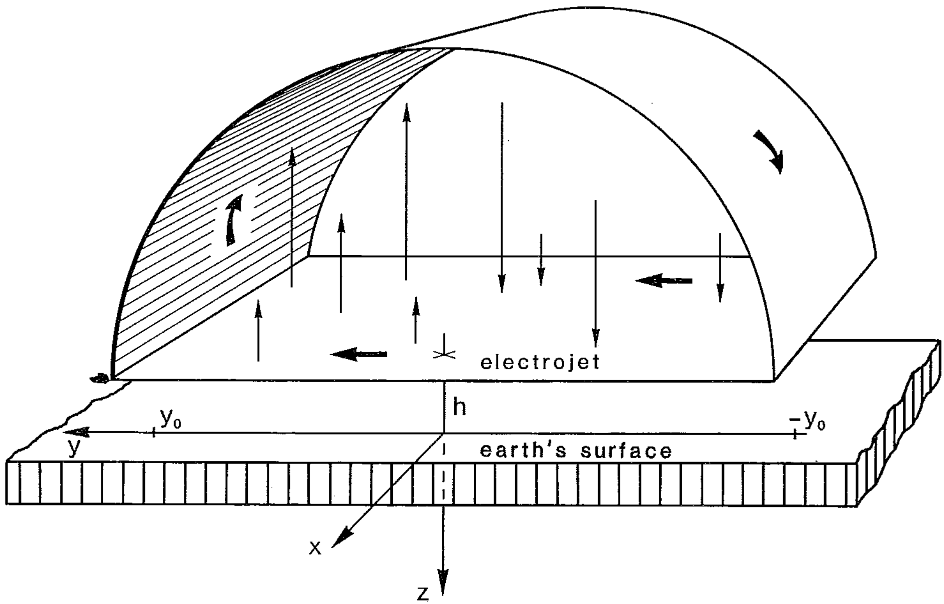

Fig. 1. Current model in which a horizontal electrojet at height h flows above a half-space earth and closes itself along the surface of a half-cylinder. Vertical currents flow within the inner region.

electromagnetic field near the earth's surface (except near »the ends of the world»). So the fields in the region $\{(x,y,z)|\ |y| \ll y_0,\ -h < z\}$ can be calculated using only equation (4) for the current density.

3. *Calculation of the primary fields*

The expression for the magnetic vector potential $\bar{A}$ in a medium with no electrical conductivity has been derived, for example, by PANOFSKY & PHILLIPS (1964, pp. 242–244) and OPPENHEIMER (1970, pp. 28–31). Applying the same method to a non-zero conductivity σ, we get

$$\bar{A}(\bar{r},t) = \frac{\mu}{4\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{V} \frac{\bar{j}(\bar{r}',\omega)e^{i(\omega t-kR)}}{R}\, d\omega d\bar{r}' \tag{5}$$

where $\bar{j}(\bar{r}',\omega)$ is the Fourier transform of the density of the primary electric current, R is defined by the equation $\bar{R} = \bar{r} - \bar{r}'$, and the complex wave number k is defined by

$$k^2 = \omega^2 \mu \epsilon - i\omega\mu\sigma \tag{6}$$

and

$$\begin{cases} -\pi/4 \leqslant \arg k < 0 \ , & \text{when } \omega > 0 \\ -\pi < \arg k < -3\pi/4 \ , & \text{when } \omega < 0 \end{cases} \tag{7}$$

So always either $Im(k) < 0$ or $k = 0$.

The following conventions for the Fourier transforms have been adopted with respect to a space coordinate and time

$$f(q,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,t) e^{iqx} dx \tag{8}$$

$$f(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,t) e^{-i\omega t} dt \tag{9}$$

So the double inverse transform has the expression

$$f(x,t) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} f(q,\omega) e^{i(\omega t - qx)} dq d\omega \tag{10}$$

Note that we use the same functional symbol f for all single and double Fourier transforms when there is no danger of confusion between the functions $f(x,t)$, $f(x,\omega)$, $f(q,t)$, and $f(q,\omega)$, which are actually different.

The magnetic flux density, usually called »the magnetic field», is the curl of the vector potential. Thus

$$\overline{B}(\overline{r},t) = \frac{\mu}{4\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_V \overline{j}(\overline{r}',\omega) \times \frac{\overline{R}}{R^2} \left(\frac{1}{R} + ik\right) e^{i(\omega t - kR)} d\omega d\overline{r}' \tag{11}$$

Let us apply this equation to the current system described in Section 2. If we express $\overline{j}(\overline{r}',\omega)$ by the double Fourier transform $J(q,\omega)$ of the total current intensity according to equation (4), we obtain

$$\begin{aligned} \overline{B}(\overline{r},t) = \frac{\mu_0}{8\pi^2} \int\int_{-\infty}^{\infty} dq d\omega J(q,\omega) e^{i\omega t} \iiint_{V(y_0)} dx' dy' dz' f(x') e^{-iqy'} e^{-ik_0 R} \cdot \\ \cdot \left(\frac{1}{R} + ik_0\right) [\delta(z'+h)\overline{e}_y - iq(1-\theta(z'+h))\overline{e}_z] \times \frac{\overline{R}}{R^2} \end{aligned} \tag{12}$$

where

$$\bar{R} = (x - x')\bar{e}_x + (y - y')\bar{e}_y + (z - z')\bar{e}_z \tag{13}$$

and the subscript $_0$ in μ_0 and k_0 refers to the air. Let y_0 grow to infinity and make the following changes in the variables: $u = x' - x$, $q_1 = y' - y$, and $\alpha = z - z'$. We also write

$$\alpha_0 = z + h \tag{14}$$

After these steps

$$\bar{B}(\bar{r},t) = \frac{\mu_0}{8\pi^2} \int\int_{-\infty}^{\infty} dq_2 d\omega J(q_2,\omega) e^{i(\omega t - q_2 y)} \bar{W}(x,z) \tag{15}$$

The function $\bar{W}$ can be divided into two parts

$$\bar{W}(x,z) = \bar{W}^1(x,z) + \bar{W}^2(x,z) \tag{16}$$

where

$$\bar{W}^1(x,z) = \int\int_{-\infty}^{\infty} du dq_1 f(u+x) \frac{e^{-ik_0\sqrt{u^2+q_1^2+\alpha_0^2}}\, e^{-iq_1q_2}}{u^2 + q_1^2 + \alpha_0^2} \cdot \cdot \left(\frac{1}{\sqrt{u^2 + q_1^2 + \alpha_0^2}} + ik_0\right)(\alpha_0\bar{e}_x + u\bar{e}_z) \tag{17}$$

and

$$\bar{W}^2(x,z) = -iq_2 \int_{\alpha_0}^{\infty} d\alpha \int\int_{-\infty}^{\infty} du dq_1 f(u+x) \frac{e^{-ik_0\sqrt{u^2+q_1^2+\alpha^2}}\, e^{-iq_1q_2}}{u^2 + q_1^2 + \alpha^2} \cdot \cdot \left(\frac{1}{\sqrt{u^2 + q_1^2 + \alpha^2}} + ik_0\right)(q_1\bar{e}_x - u\bar{e}_y) \tag{18}$$

For the moment we assume that ω and q_2 are not simultaneously zero, and we introduce a new function L with the equation

$$L(u,\alpha) \equiv \int_{-\infty}^{\infty} \frac{e^{-ik_0\sqrt{u^2+q_1^2+\alpha^2}}}{\sqrt{u^2 + q_1^2 + \alpha^2}} e^{-iq_1q_2} dq_1 \tag{19}$$

The following expressions for $\bar{W}^1$ and $\bar{W}^2$ are now easy to obtain in the region where $\alpha \geqslant \alpha_0 > 0$ (or $z > -h$)

$$\bar{W}^1(x,z) = -\bar{e}_x \int_{-\infty}^{\infty} f(u+x) \frac{\partial}{\partial \alpha_0} L(u,\alpha_0) du + \bar{e}_z \int_{-\infty}^{\infty} L(u,\alpha_0) f'(u+x) du \quad (20)$$

$$\bar{W}^2(x,z) = iq_2 \int_{\alpha_0}^{\infty} d\alpha [iq_2 \bar{e}_x \int_{-\infty}^{\infty} f(u+x) L(u,\alpha) du + \bar{e}_y \int_{-\infty}^{\infty} L(u,\alpha) f'(u+x) du] \quad (21)$$

Using symmetry we can also write

$$L(u,\alpha) = 2K_0(\sqrt{(-u)^2+\alpha^2} \cdot \sqrt{q_2^2+(ik_0)^2}) = \int_{-\infty}^{\infty} dq_1 \frac{e^{-\alpha\sqrt{q_1^2+q_2^2-k_0^2}}}{\sqrt{q_1^2+q_2^2-k_0^2}} e^{iuq_1} \quad (22)$$

where, for the sake of uniqueness and convergence, we have the restriction $Re(\sqrt{q_1^2+q_2^2-k_0^2}) > 0$. So if we utilize the Fourier transform $f(q_1)$ of the density function $f(x)$ and define the vector $\bar{q}$ by $\bar{q} = q_1\bar{e}_x + q_2\bar{e}_y$, the functions $\bar{W}^1$ and $\bar{W}^2$ can be written as

$$\bar{W}^1(x,z) = \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-(ixq_1+\alpha_0\sqrt{q^2-k_0^2})} f(q_1)\left(\bar{e}_x - \frac{iq_1\bar{e}_z}{\sqrt{q^2-k_0^2}}\right) dq_1 \quad (23)$$

$$\bar{W}^2(x,z) = -q_2\sqrt{2\pi} \int_{\alpha_0}^{\infty} d\alpha \int_{-\infty}^{\infty} \frac{e^{-(ixq_1+\alpha\sqrt{q^2-k_0^2}}}{\sqrt{q^2-k_0^2}} f(q_1)(q_2\bar{e}_x - q_1\bar{e}_y) dq_1$$

$$= -q_2\sqrt{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(ixq_1+\alpha_0\sqrt{q^2-k_0^2})}}{q^2-k_0^2} f(q_1)(q_2\bar{e}_x - q_1\bar{e}_y) dq_1 \quad (24)$$

The case $\omega = 0$ and $q_2 = 0$ can be calculated directly from equations (17) and (18). However, the calculations lead to the same formulas as those obtained from equations (23) and (24) in the limit $\omega \to 0$ and $q_2 \to 0$.

Finally, the expression for the magnetic flux density is

$$\bar{B}(\bar{r},t) = \frac{\mu_0}{2(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \int\int_{-\infty}^{\infty} dq_1 dq_2 J(q_2,\omega) e^{-(i\bar{q}\cdot\bar{r}+\alpha_0\sqrt{q^2-k_0^2})} \cdot$$

$$\cdot f(q_1)\left[\left(1 - \frac{q_2^2}{q^2-k_0^2}\right)\bar{e}_x + \frac{q_1 q_2 \bar{e}_y}{q^2-k_0^2} - \frac{iq_1\bar{e}_z}{\sqrt{q^2-k_0^2}}\right] \quad (25)$$

The expression in the square brackets remains bounded near the point where $q^2 - k_0^2 = 0$. So this point does not cause problems in calculating the triple integral.

Let us assume that the air below the ionosphere is a simple medium in which

Ohm's law is valid. The fourth Maxwell's equation will then be

$$\nabla \times \bar{B} = \mu_0 \bar{j}_{sec} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t} = \mu_0 \sigma_0 \bar{E} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t} \tag{26}$$

or

$$\nabla \times \bar{B}(\bar{r}, \omega) = \mu_0 (\sigma_0 + i\omega\epsilon_0) \bar{E}(\bar{r}, \omega) \tag{27}$$

Hence the Fourier transform $\bar{E}(\bar{r}, \omega)$ can be calculated using equation (25). The electric field can then be expressed as follows

$$\bar{E}(\bar{r}, t) = -\frac{\mu_0}{2(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\omega\, \omega e^{i\omega t} \int\int_{-\infty}^{\infty} dq_1\, dq_2\, J(q_2, \omega) e^{-(i\bar{q}\cdot\bar{r} + \alpha_0 \sqrt{q^2 - k_0^2})} \cdot$$

$$\cdot f(q_1) \left[\frac{i\bar{e}_y}{\sqrt{q^2 - k_0^2}} + \frac{q_2 \bar{e}_z}{q^2 - k_0^2} \right] \tag{28}$$

4. *Effect of geomagnetic induction*

We assume that the earth is composed of two uniform horizontal layers, the first of which lies in the region where $0 \leqslant z \leqslant z_1$ $(z_1 > 0)$ and the other in the region where $z_1 < z < \infty$ (Figure 2). The permittivity, permeability, and conductivity of the j:th layer are denoted by ϵ_j, μ_j, and σ_j $(j = 1, 2)$. These quantities are assumed to be positive constants.

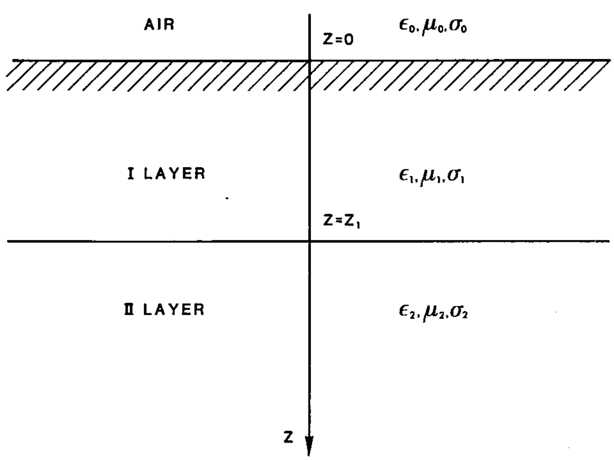

Fig. 2. Model of the earth composed of two horizontal layers.

The wave equations for the magnetic flux density and the electric field in the earth (j = 1, 2) and in the air (j = 0) are derived directly from the Maxwell's curl equations. They are

$$\nabla^2\bar{B} - \mu_j\left(\sigma_j \frac{\partial \bar{B}}{\partial t} + \epsilon_j \frac{\partial^2 \bar{B}}{\partial t^2}\right) = 0 \tag{29}$$

$$\nabla^2\bar{E} - \mu_j\left(\sigma_j \frac{\partial \bar{E}}{\partial t} + \epsilon_j \frac{\partial^2 \bar{E}}{\partial t^2}\right) = 0 \tag{30}$$

The Fourier transformation of equation (30) is

$$\frac{\partial^2 \bar{E}(x, q_2, z, \omega)}{\partial x^2} + \frac{\partial^2 \bar{E}(x, q_2, z, \omega)}{\partial z^2} + \eta_j^2 \bar{E}(x, q_2, z, \omega) = 0 \tag{31}$$

where η_j is defined by the formulas

$$\begin{cases} \eta_j^2 = k_j^2 - q_2^2 = \omega^2\mu_j\epsilon_j - q_2^2 - i\omega\mu_j\sigma_j \\ 0 \leqslant \arg \eta_j < \pi \end{cases} \tag{32}$$

It is impossible for $\arg\eta_j$ to equal 0, so either $Im(\eta_j) > 0$ or $\eta_j = 0$. A particular solution for the component $E_y(x, q_2, z, \omega)$ can be obtained by separation of variables and the general solution is formed by integrating the particular solution over all the real values of the separation parameter:

$$E_y^0(x, q_2, z, \omega) = \int_{-\infty}^{\infty} D_0(q_1, q_2, \omega) e^{\xi_0 z} e^{-iq_1 x} dq_1 \tag{33}$$

$$E_y^1(x, q_2, z, \omega) = \int_{-\infty}^{\infty} [D_1(q_1, q_2, \omega) e^{\xi_1 z} + G_1(q_1, q_2, \omega) e^{-\xi_1 z}] e^{-iq_1 x} dq_1 \tag{34}$$

$$E_y^2(x, q_2, z, \omega) = \int_{-\infty}^{\infty} G_2(q_1, q_2, \omega) e^{-\xi_2 z} e^{-iq_1 x} dq_1 \tag{35}$$

where the superscript denotes the layer in question. The functions ξ_j (j = 0, 1, 2) are defined by the formulas

$$\begin{cases} \xi_j^2 = q_1^2 - \eta_j^2 = q^2 - k_j^2 \\ -\pi/2 < \arg \xi_j \leqslant \pi/2 \end{cases} \tag{36}$$

and so $Re(\xi_j) > 0$ or $\xi_j = 0$. By analogy, we can derive the following from equation (29)

$$B_y^0(x, q_2, z, \omega) = \int_{-\infty}^{\infty} Q_0(q_1, q_2, \omega) e^{\xi_0 z} e^{-iq_1 x} dq_1 \tag{37}$$

$$B_y^1(x, q_2, z, \omega) = \int_{-\infty}^{\infty} [Q_1(q_1, q_2, \omega) e^{\xi_1 z} + R_1(q_1, q_2, \omega) e^{-\xi_1 z}] e^{-iq_1 x} dq_1 \tag{38}$$

$$B_y^2(x, q_2, z, \omega) = \int_{-\infty}^{\infty} R_2(q_1, q_2, \omega) e^{-\xi_2 z} e^{-iq_1 x} dq_1 \tag{39}$$

To enable the other components to be expressed using the functions $D_0(q_1, q_2, \omega)$, ..., $R_2(q_1, q_2, \omega)$, we write the x and z-components of the Maxwell' curl equations for the quantities $\bar{E}(x, q_2, z, \omega)$ and $\bar{B}(x, q_2, z, \omega)$ and solve the equations for $E_x(x, q_2, z, \omega)$, $E_z(x, q_2, z, \omega)$, $B_x(x, q_2, z, \omega)$, and $B_z(x, q_2, z, \omega)$. The results are

$$E_x^j(x, q_2, z, \omega) = \frac{1}{\eta_j^2} \left[-iq_2 \frac{\partial E_y^j(x, q_2, z, \omega)}{\partial x} + i\omega \frac{\partial B_y^j(x, q_2, z, \omega)}{\partial z} \right] \tag{40}$$

$$E_z^j(x, q_2, z, \omega) = -\frac{1}{\eta_j^2} \left[iq_2 \frac{\partial E_y^j(x, q_2, z, \omega)}{\partial z} + i\omega \frac{\partial B_y^j(x, q_2, z, \omega)}{\partial x} \right] \tag{41}$$

$$B_x^j(x, q_2, z, \omega) = -\frac{1}{\eta_j^2} \left[\mu_j(\sigma_j + i\omega\epsilon_j) \frac{\partial E_y^j(x, q_2, z, \omega)}{\partial z} + iq_2 \frac{\partial B_y^j(x, q_2, z, \omega)}{\partial x} \right] \tag{42}$$

$$B_z^j(x, q_2, z, \omega) = \frac{1}{\eta_j^2} \left[\mu_j(\sigma_j + i\omega\epsilon_j) \frac{\partial E_y^j(x, q_2, z, \omega)}{\partial x} - iq_2 \frac{\partial B_y^j(x, q_2, z, \omega)}{\partial z} \right] \tag{43}$$

The above expressions for the components of the electric field and the magnetic flux density in the layers 1 and 2 describe the total field inside the earth. To get the total field in the air (region 0), we have to sum the primary fields (equations (25) and (28)) and the secondary fields $\bar{E}^0$ and $\bar{B}^0$.

The continuity of the tangential components E_x, E_y, B_x/μ, and B_y/μ at the boundary surfaces of the different layers is used to determine the functions D_0, Q_0, D_1, G_1, Q_1, R_1, G_2 and R_2. The outlines of the procedure will be found in PIRJOLA (1982). Here we merely quote the results:

$$D_0(q_1, q_2, \omega) = \frac{i\mu_0\omega\eta_0^2\eta_1^2}{\xi_0 M} \left[\frac{M}{2\eta_0^2\eta_1^2} - \frac{\mu_1}{\mu_0} \xi_0\xi_1 k_0^2 A_1 + (q_1^2 k_1^2 - k_0^2\eta_1^2) A_0 + \right.$$

$$\left. + 2iq_1 q_2 \xi_1 \frac{k_1^2 \alpha_r}{\omega} \right] \frac{J(q_2, \omega) f(q_1) e^{-h\xi_0}}{\sqrt{2\pi}} \tag{44}$$

$$Q_0(q_1,q_2\,\omega) = -\frac{\mu_0 q_1 q_2 \eta_0^2\eta_1^2}{\xi_0 M}\left(\frac{M}{2\xi_0\eta_0^2\eta_1^2} - \frac{\mu_0}{\mu_1}\,\xi_1 k_1^2 A_2 + \xi_0 k_0^2 A_0 + \right.$$

$$\left. - \frac{2i\omega\xi_0\xi_1 k_0^2\beta_g}{q_1 q_2}\right)\frac{J(q_2,\omega)f(q_1)e^{-h\xi_0}}{\sqrt{2\pi}} \tag{45}$$

where

$$M = [q_1^2q_2^2(k_1^2 - k_0^2)^2 - k_0^2\xi_0^2\eta_1^4]A_0 - 2iq_1q_2\xi_1\eta_0^2(k_1^2 - k_0^2)\left(\frac{\alpha_r k_1^2}{\omega} - \omega\beta_g\right) +$$

$$+ \xi_0\xi_1\eta_0^2\eta_1^2\left(\frac{\mu_0}{\mu_1}\,k_1^2A_2 + \frac{\mu_1}{\mu_0}\,k_0^2A_1\right) - k_1^2\xi_1^2\eta_0^4A_3 \tag{46}$$

and

$$\begin{cases} A_0 = (\alpha_g + 1)(\beta_r + 1) - \alpha_r\beta_g \\ A_1 = (\alpha_g + 1)(\beta_r - 1) - \alpha_r\beta_g \\ A_2 = (\alpha_g - 1)(\beta_r + 1) - \alpha_r\beta_g \\ A_3 = (\alpha_g - 1)(\beta_r - 1) - \alpha_r\beta_g \end{cases} \tag{47}$$

The functions α_g, α_r, β_g, and β_r are defined by the equations

$$D_1(q_1,q_2,\omega) = \alpha_g(q_1,q_2,\omega)G_1(q_1,q_2,\omega) + \alpha_r(q_1,q_2,\omega)R_1(q_1,q_2,\omega) \tag{48}$$

$$Q_1(q_1,q_2,\omega) = \beta_g(q_1,q_2,\omega)G_1(q_1,q_2,\omega) + \beta_r(q_1,q_2,\omega)R_1(q_1,q_2,\omega) \tag{49}$$

and they are

$$\alpha_g(q_1,q_2,\omega) = \frac{e^{-2\xi_1 z_1}}{L}\left[-q_1^2q_2^2(k_2^2 - k_1^2)^2 - k_1^2\xi_1^2\eta_2^4 + k_2^2\xi_2^2\eta_1^4 + \right.$$

$$\left. - \frac{\mu_2}{\mu_1}\,k_1^2\xi_1\xi_2\eta_1^2\eta_2^2 + \frac{\mu_1}{\mu_2}\,k_1^2\xi_1\xi_2\eta_1^2\eta_2^2\right] \tag{50}$$

$$\alpha_r(q_1,q_2,\omega) = -\frac{2e^{-2\xi_1 z_1}}{L}\,i\omega q_1q_2(k_2^2 - k_1^2)\xi_1\eta_2^2 \tag{51}$$

$$\beta_g(q_1,q_2,\omega) = -\frac{2e^{-2\xi_1 z_1}}{i\omega L}\, q_1 q_2(k_2^2 - k_1^2)k_1^2\xi_1\eta_2^2 \tag{52}$$

$$\beta_r(q_1,q_2,\omega) = \frac{e^{-2\xi_1 z_1}}{L}\Big[-q_1^2q_2^2(k_2^2-k_1^2)^2 - k_1^2\xi_1^2\eta_2^4 + k_2^2\xi_2^2\eta_1^4 + + \frac{\mu_2}{\mu_1}k_1^2\xi_1\xi_2\eta_1^2\eta_2^2 - \frac{\mu_1}{\mu_2}k_2^2\xi_1\xi_2\eta_1^2\eta_2^2\Big] \tag{53}$$

where

$$L = q_1^2q_1^2(k_2^2-k_1^2)^2 - k_1^2\xi_1^2\eta_2^4 - k_2^2\xi_2^2\eta_1^4 - \frac{\mu_2}{\mu_1}k_1^2\xi_1\xi_2\eta_1^2\eta_2^2 - \frac{\mu_1}{\mu_2}k_2^2\xi_1\xi_2\eta_1^2\eta_2^2 \tag{54}$$

The functions α_g and β_r are dimensionless, whereas α_r and β_g are in ms^{-1} and $m^{-1}s$, respectively.

The limit value of the electric field on the earth's surface, when approached from above ($z \to 0-$), is now calculated by summing the primary field (equation (28)) and the secondary field (obtainable from equations (33), (40), (41), (44), and (45)) and setting $z = 0$. In the same way, the limit $z \to 0-$ of the magnetic flux density is calculated from equations (25), (37), (42), (43), (44), and (45). When the Fourier components are integrated, we obtain

$$E_x(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\Big\{-i\omega q_1q_2\Big[(q_1^2k_1^2-\eta_1^2k_0^2)\Big(A_0 - \frac{2i\xi_1\omega\beta_g}{q_1q_2}\Big) + - k_1^2\xi_1^2A_3 + 2iq_1q_2\xi_1\frac{\alpha_r k_1^2}{\omega}\Big]\Big\} \tag{55}$$

$$E_y(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\Big\{i\omega\eta_1^2\Big[-\frac{\mu_1}{\mu_0}\xi_0\xi_1k_0^2A_1 + + (q_1^2k_1^2 - k_0^2\eta_1^2)A_0 + 2iq_1q_2\xi_1\frac{k_1^2\alpha_r}{\omega}\Big]\Big\} \tag{56}$$

$$E_z(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\Big\{\omega q_2k_1^2\xi_0\Big(-\frac{\mu_0\xi_1}{\mu_1\xi_0}\eta_1^2A_2 + 2iq_1q_2\frac{\xi_1\alpha_r}{\omega} + - 2i\frac{q_1}{q_2}\xi_1\omega\beta_g - \xi_1^2A_3 + q_1^2A_0\Big)\Big\} \tag{57}$$

$$B_x(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\left\{\xi_0 k_0^2\left[q_1^2 q_2^2 A_0 + \frac{\mu_0 \xi_1 k_1^2}{\mu_1 \xi_0 k_0^2}\,\eta_1^2(q_1^2 - k_0^2)A_2 + \right.\right.$$
$$\left.\left. - k_1^2 \xi_1^2 A_3 + 2iq_1 q_2 \xi_1\left(\frac{k_1^2 \alpha_r}{\omega} - \omega\beta_g\right)\right]\right\} \tag{58}$$

$$B_y(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\left\{q_1 q_2 \eta_1^2\left(\frac{\mu_0}{\mu_1}\,\xi_1 k_1 A_2 - \xi_0 k_0^2 A_0 + \right.\right.$$
$$\left.\left. + \frac{2i\xi_0 \xi_1 k_0^2 \omega\beta_g}{q_1 q_2}\right)\right\} \tag{59}$$

$$B_z(x,y,z{=}0,t) = \hat{N}(x,y,t|q_1,q_2,\omega)\left\{iq_1\left[(k_1^2 q_1^2 - k_0^2\eta_1^2)\left(k_1^2 A_0 - \frac{2iq_2\xi_1\omega\beta_g}{g_1}\right) + \right.\right.$$
$$\left.\left. - \frac{\mu_1}{\mu_0}\,\xi_0\xi_1\eta_1^2 k_0^2 A_1 - q_2^2 k_1^2 \xi_1^2 A_3 + 2iq_1 q_2 \xi_1 k_1^2\,\frac{\alpha_r k_1^2}{\omega}\right]\right\} \tag{60}$$

where the integral operator $\hat{N}$ is defined by

$$\hat{N}(x,y,t|q_1,q_2,\omega)\,\psi(q_1,q_2\;\omega) \equiv$$

$$\equiv \frac{\mu_0}{(2\pi)^{3/2}} \int\int_{-\infty}^{\infty}\int d\omega dq_1 dq_2 e^{i(\omega t - \bar{q}\cdot\bar{r})} e^{-h\xi_0} J(q_2,\omega) f(q_1)\,\frac{\eta_0^2}{\xi_0 M}\,\psi(q_1,q_2,\omega) \tag{61}$$

Equations (55) to (60) for the electromagnetic field satisfy the wave equations (29) and (30) when the z-dependence is retained. The latter equations were derived from the Maxwell's equations, but not every solution $\bar{E},\bar{B}$ of the wave equations satisfies the Maxwell's equations. A direct calculation shows, however, that the fields given by equations (55) to (60), with the z-dependence, satisfy also the Maxwell's equations, thus providing the correct solutions.

5. *Special cases*

If the current intensity of the electrojet does not depend on the y-coordinate, *i.e.*

$$\frac{\partial J(y,t)}{\partial y} = 0 \tag{62}$$

the double Fourier transform is

$$J(q_2,\omega) = \sqrt{2\pi}\, J_1(\omega)\delta(q_2) \tag{63}$$

where

$$J_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} J(y,t)e^{-i\omega t}dt \tag{64}$$

In this case there are no vertical currents, and the expressions for the fields are reduced to the short forms

$$\bar{E}(x,z{=}0,t) = -\frac{i\mu_0\bar{e}_y}{2\pi} \int\int_{-\infty}^{\infty} J_1(\omega)f(q_1)e^{-h\xi_0}e^{i(\omega t-q_1x)}\frac{\omega}{\xi_0(1+\phi^{-1})}\, d\omega dq_1 \tag{65}$$

$$\bar{B}(x,z{=}0,t) = \frac{\mu_0}{2\pi} \int\int_{-\infty}^{\infty} J_1(\omega)f(q_1)e^{-h\xi_0}e^{i(\omega t-q_1x)}\left[\frac{\bar{e}_x}{1+\phi} - \frac{iq_1\bar{e}_z}{\xi_0(1+\phi^{-1})}\right] d\omega dq_1 \tag{66}$$

where

$$\phi(q_1,\omega) \equiv \frac{\mu_1\xi_0}{\mu_0\xi_1}\,\frac{\mu_1\xi_1A_1k_0^2-\mu_0\xi_0A_0k_1^2}{\mu_0\xi_0A_2k_1^2-\mu_1\xi_1A_3k_0^2} = \frac{\mu_1\xi_0}{\mu_0\xi_1}\,\frac{1+\alpha_g(q_1,q_2{=}0,\omega)}{1-\alpha_g(q_1,q_2{=}0,\omega)} \tag{67}$$

and

$$\alpha_g(q_1,q_2{=}0,\omega) = \frac{\dfrac{\xi_1}{\mu_1}-\dfrac{\xi_2}{\mu_2}}{\dfrac{\xi_1}{\mu_1}+\dfrac{\xi_2}{\mu_2}}\, e^{-2\xi_1z_1} \tag{68}$$

If the current intensity does not depend on time, *i.e.*

$$\frac{\partial J(y,t)}{\partial t} = 0 \tag{69}$$

we obtain

$$J(q_2,\omega) = \sqrt{2\pi}\, J_2(q_2)\delta(\omega) \tag{70}$$

where

$$J_2(q_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} J(y,t)e^{iyq_2}dy \tag{71}$$

After some calculation we obtain

$$\bar{E}(x,y) = 0 \tag{72}$$

$$B_x(x,y) = \frac{\mu_0}{2\pi} \int\int_{-\infty}^{\infty} dq_1 dq_2 e^{-i\bar{q}\cdot\bar{r}} e^{-hq} J_2(q_2) f(q_1) \frac{q_1^2/q^2}{1 + \frac{\mu_1}{\mu_0}\frac{1-\beta_r}{1+\beta_r}} \tag{73}$$

$$B_y(x,y) = \frac{\mu_0}{2\pi} \int\int_{-\infty}^{\infty} dq_1 dq_2 e^{-i\bar{q}\cdot\bar{r}} e^{-hq} J_2(q_2) f(q_1) \frac{q_1 q_2/q^2}{1 + \frac{\mu_1}{\mu_0}\frac{1-\beta_r}{1+\beta_r}} \tag{74}$$

$$B_z(x,y) = \frac{\mu_0}{2\pi} \int\int_{-\infty}^{\infty} dq_1 dq_2 e^{-i\bar{q}\cdot\bar{r}} e^{-hq} J_2(q_2) f(q_1) \frac{-iq_1/q}{1 + \frac{\mu_0}{\mu_1}\frac{1+\beta_r}{1-\beta_r}} \tag{75}$$

where β_r is now simply

$$\beta_r = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} e^{-2qz_1} \tag{76}$$

If the magnetic structure of the earth is homogeneous, then $\mu_1 = \mu_2$, and so $\beta_r = 0$. The previous equations now give

$$\bar{B}(x,y) = \frac{\mu_0}{2\pi(\mu_0 + \mu_1)} \int\int_{-\infty}^{\infty} dq_1 dq_2 e^{-i\bar{q}\cdot\bar{r}} e^{-hq} J_2(q_2) f(q_1) \frac{q_1}{q} \left(\frac{\mu_0}{q}\bar{q} - i\mu_1 \bar{e}_z\right) \tag{77}$$

If there is neither y- nor time dependence and $\mu_1 = \mu_2$, we put

$$J_2(q_2) = \sqrt{2\pi}\, J\delta(q_2) \tag{78}$$

in equation (77) and get

$$\bar{B}(x) = \frac{\mu_0 J}{\sqrt{2\pi}(\mu_0 + \mu_1)} \int_{-\infty}^{\infty} dq_1 e^{-iq_1 x} e^{-h|q_1|} f(q_1) \left(\mu_0 \bar{e}_x - i\mu_1 \frac{q_1}{|q_1|} \bar{e}_z\right) =$$

$$= \frac{\mu_0 J}{\pi(\mu_0 + \mu_1)} \int_{-\infty}^{\infty} dx' f(x') [\mu_0 \bar{e}_x Re(I(x,x')) + \mu_1 \bar{e}_z Im(I(x,x'))] \tag{79}$$

where

$$I(x,x') \equiv \int_0^\infty e^{[i(x'-x)-h]q_1} dq_1 = \frac{1}{h - i(x'-x)} \tag{80}$$

Thus the magnetic flux density is

$$\bar{B}(x) = \frac{\mu_0 J}{\pi(\mu_0 + \mu_1)} \int_{-\infty}^{\infty} \frac{\mu_0 h \bar{e}_x + \mu_1 (x'-x)\bar{e}_z}{(x'-x)^2 + h^2} f(x')dx' \tag{81}$$

6. *Discussion and conclusions*

A theoretical model for the high-latitude ionospheric large-scale electric current (the electrojet current system) is presented above and the corresponding electromagnetic field is calculated. The model had to include simplifying assumptions on the shape and structure of the earth to allow the treatment of geomagnetic induction. The characteristics of the model are:

1. The earth is described as a half-space with an infinite plane surface. It consists of two uniform horizontal layers, within which the permittivity, permeability, and conductivity are constant.
2. The permittivity and permeability have their vacuum values in the air below the ionosphere. The air is assumed to have a low conductivity but no primary charge, and no electric currents apart from those explicitly stated below.
3. The most important part of the electrojet current system is an infinitely thin horizontal sheet current. Its current intensity has an arbitrary dependence on time and on the space coordinate in the direction of the current flow. The current density has an arbitrary distribution in the direction perpendicular to the flow. This cross-sectional distribution is constant in time and in the direction of the flow.
4. Vertical currents above the horizontal jet make the divergence of the total current density to vanish.

This model covers a wide variety of electrojet current forms and time developments. The accumulation of charge in the ionosphere is avoided, thanks to the vertical currents.

In addition to the assumption of a flat earth, the model has the following limitation: the electrojet is assumed to have the same cross-sectional distribution at every moment of time and in every cross section along the jet. According to GRAFE (1983), the width of the electrojet may change considerably even in ten minutes. So a more general current distribution model could be useful in special cases. But even then a superposition of the current systems studied here may provide a good approximation.

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