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APPLICATION OF A MATHEMATICAL MODEL TO A BRANCHED WATERCOURSE

by

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Abstract

A one-dimensional mathematical model is applied to a branched natural watercourse, using an implicit finite difference scheme for the governing equations. The use of generalized expressions for the boundary conditions for practical applications is explained. The calculated and measured discharges close to the junction point are compared, and are found to agree fairly well with each other.

1. *Introduction*

Mathematical modelling of flow in watercourses has become more and more important as a result of the development of models and computers. Natural watercourses are almost always quite irregular with changing cross sections and bottom topography along their main axis, which makes it impossible to solve the governing hydrodynamic equations with sufficient accuracy by traditional analytical means. This becomes even more difficult if the watercourse is diverging in several branches, thus increasing the complexity of water flow. However, sometimes there is a need to predict the flow in complex watercourses. Such is the case when major water works are planned which may affect the environment in a non-desired way.

In this article we will present the application of a one-dimensional mathematical model to a branched natural watercourse. The diurnal regulation of the discharge from a hydropower plant influences the distribution of flow between two branches in the watercourse with subsequent influence on the water quality in the branches. Special attention has been paid to the formulation of boundary conditions in the model.

2. *Equations*

The hydrodynamic equations describing one-dimensional unsteady open channel flow are usually given in the form of the de St Venant equations:

$$b_s \frac{\partial z}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{2.1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA \frac{Q \cdot Q}{K^2} = 0 \tag{2.2}$$

Here Q is the discharge, z the water level measured from a horizontal datum, b_s the width available for storage, q the lateral inflow, A the cross-sectional flow area, K the conveyance, g the acceleration of gravity and β is a coefficient for the non-uniform velocity distribution.

After discretization, the above equations can be solved numerically when given one initial condition and two boundary conditions. Usually the physical coefficients b_s, A, K and to some extent β vary with stage, giving the equations a non-linear character. The coefficients can most conveniently be tabulated as functions of z, from which coefficient values corresponding to the current water level are obtained by linear interpolation.

3. *The difference scheme and the solution algorithm*

The equations (2.1) and (2.2) can be discretized in a number of ways. In this case a four-point scheme of Preissmann type has been used. According to this the partial derivatives are approximated by

$$\frac{\partial f}{\partial t} = \psi \frac{f_{j+1}^{n+1} - f_{j+1}^{n}}{\Delta t} + (1-\psi) \cdot \frac{f_j^{n+1} - f_j^n}{\Delta t} \tag{3.1}$$

$$\frac{\partial f}{\partial x} = \Theta \cdot \frac{f_{j+1}^{n+1} - f_j^{n+1}}{\Delta x} + (1-\Theta) \cdot \frac{f_{j+1}^{n} - f_j^n}{\Delta x} \tag{3.2}$$

where f is either Q or z, n is the time level, j the computational point, ψ and Θ are weighting coefficients.

In the fully centred case ($\psi = \Theta = 0.5$) the degree of approximation of both time and space derivatives is of second order. Thus the difference scheme used is the following when $\psi = 0.5$

$$b_{s_{j+1/2}}^{n+\Theta}\left[\frac{1}{2}\,\frac{z_j^{n+1}-z_j^n}{\Delta t}+\frac{1}{2}\,\frac{z_{j+1}^{n+1}-z_{j+1}^n}{\Delta t}\right]+\Theta\,\frac{Q_{j+1}^{n+1}-Q_j^{n+1}}{\Delta x}+$$

$$+(1-\Theta)\,\frac{Q_{j+1}^n-Q_j^n}{\Delta x}-q_{j+1/2}^{n+\Theta}=0 \qquad (3.3)$$

$$\frac{1}{2}\,\frac{Q_j^{n+1}-Q_j^n}{\Delta t}+\frac{1}{2}\,\frac{Q_{j+1}^{n+1}-Q_{j+1}^n}{\Delta t}+\frac{\left(\frac{\beta}{A}\right)_{j+1}^{n+1/2}Q_{j+1}^nQ_{j+1}^{n+1}-\left(\frac{\beta}{A}\right)_j^{n+1/2}Q_j^nQ_j^{n+1}}{\Delta x}+$$

$$+gA_{j+1/2}^{n+\Theta}\left[\Theta\,\frac{z_{j+1}^{n+1}-z_j^{n+1}}{\Delta x}-(1-\Theta)\,\frac{z_{j+1}^n-z_j^n}{\Delta x}\right]+$$

$$\frac{1}{2}\,gA_{j+1/2}^{n+1/2}\left[\frac{|Q_j^n|\cdot Q_j^{n+1}}{(K^2)_j^{n+1/2}}+\frac{|Q_{j+1}^n|\cdot Q_{j+1}^{n+1}}{(K^2)_{j+1}^{n+1/2}}\right]=0 \qquad (3.4)$$

The non-linear term $(\partial/\partial x)(\beta Q^2/A)$ in eq. (2.2) has been discretized according to a method following Verwey (CUNGE *et al.*, 1980). The difference equations (3.3) and (3.4) are formally of the type

$$A1_jQ_j^{n+1}+B1_jz_j^{n+1}+C1_jQ_{j+1}^{n+1}+D1_jz_{j+1}^{n+1}=E1_j \qquad (3.5)$$

$$A2_jQ_j^{n+1}+B2_jz_j^{n+1}+C2_jQ_{j+1}^{n+1}+D2_jz_{j+1}^{n+1}=E2_j \qquad (3.6)$$

Thus the scheme is implicit and for large models (models with many computational points) a double sweep procedure is most economical, and is used here. Double sweep methods are necessary only for subcritical flows. A description of the procedure required is given *e.g.* by ABBOTT (1979), but because the application of the method to branched systems requires special treatment of junction points, a short description will be given here. The explanation of the numerical treatment of the boundary conditions is also aided through this. The following relations are introduced, dropping the superscript:

$$Q_j=F_jz_j+G_j \qquad (3.7)$$

$$z_j=H_jQ_{j+1}+I_jz_{j+1}+J_j \qquad (3.8)$$

Algebraic expressions for the coefficients H_j, I_j and J_j are obtained after inserting equation (3.8) into equation (3.5). Recurrence relations for F_j and G_j are obtained after substitution of equations (3.7) and (3.8) into equation (3.6).

Thus $F_{j+1} = f(F_j)$ and $G_{j+1} = f(F_j, G_j)$. After initialization of F_1 and G_1 in accordance with the boundary conditions in the point $j = 1$, the first sweep can be carried out from point $j = 1$ to $j = jj - 1$, where jj is the last grid point. In this sweep values for F_{j+1}, G_{j+1}, H_j, I_j and J_j are computed. In point jj the second boundary condition is applied, and the return sweep is executed using equations (3.7) and (3.8) for the computation of Q- and z-values in internal gridpoints. The flow direction relative to the sweep directions is of no importance, the final result will be the same.

4. *Boundary conditions*

4.1 Single channel

In models intended for routine applications it is convenlent to use general expressions for the boundary conditions, which then can be expressed with coefficients α, β and γ as:

$$\alpha Q^{n+1} + \beta z^{n+1} = \gamma \tag{4.1}$$

In the first gridpoint the initialization of F and G is obtained through

$$F_1 = -\frac{\beta_1}{\alpha_1}; \quad G_1 = \frac{\gamma_1}{\alpha_1} \tag{4.2}$$

In the last gridpoint one can introduce the additional relation:

$$z_{jj} = \frac{\gamma_{jj} - \alpha_{jj} G_{jj}}{\beta_{jj} + \alpha_{jj} F_{jj}} \tag{4.3}$$

Values for the coefficients are determined from the boundary conditions which usually are of the following type:

1) Q given as a function of time (*e.g.* the regulated flow from a hydropower plant).
2) z given as a function of time (*e.g.* the water level of a reservoir).
3) A $Q - z$ relationship in the form of a rating curve (*e.g.* the free flow over a crest or a weir)

In the above cases the values of α, β and γ are obtained as follows.

1) Discharge is given:

$$\alpha = 1\,; \qquad \beta = 0\,; \qquad \gamma = Q_1^{n+1} \quad \text{or} \quad = Q_{jj}^{n+1}$$

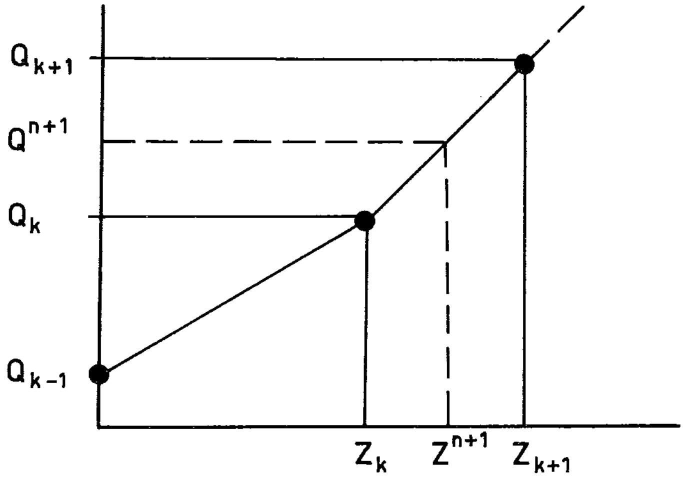

Fig. 1. Discretization of a stage-discharge relationship.

2) Water level is given:

$$\alpha_1 = 10^{-4}...10^{-6}; \quad \beta_1 = 1\ ; \quad \gamma_1 = z_j^{n+1}$$

$$\alpha_{jj} = 0 \quad \beta_{jj} = 1\ ; \quad \gamma_{jj} = z_{jj}^{n+1}$$

A small value of α_1 is required in this case to avoid division by zero in equation (4.2). The thus introduced error is usually negligible in practice.

3) A rating curve is given.
If the stage-discharge relationship is given as a tabulated function, it can, using the notation in Fig. 1, be expressed as

$$Q^{n+1} = Q_k + \frac{dQ}{dz}\,(z^{n+1} - z_k)$$

Thus we obtain

$$\alpha = 1\ ; \quad \beta = \frac{dQ}{dz}\ ; \quad \gamma = Q_k - \frac{dQ}{dz}\, z_k$$

A stage-discharge relationship cannot be applied at the upstream boundary, because it will lead to numerical instability (CUNGE *et al.* 1980).

4.2 Internal boundary conditions

In branched watercourses the junctions are points which require special treatment, when applying double sweep methods. One internal boundary condition that links all the different branches in a junction to each other has to be introduced. Let us consider a watercourse which is discretized as shown in Fig. 2. The first sweep can be initiated at ① using the applied boundary condition and can be carried on to point 11. Another sweep can be initiated at ② and carried on to point 21. But before one continues sweep to point ③, points 11, 21 and 31 has to be linked together assuming compatibility conditions in these points. From continuity one gets

$$Q_{11} + Q_{21} = Q_{31} \tag{4.4}$$

and assuming water level compatibility

$$z_{11} = z_{21} = z_{31} \tag{4.5}$$

the following explicit expressions can be obtained:

$$F_{31} = F_{11} + F_{21} \tag{4.6}$$

$$G_{31} = G_{11} + G_{21} \tag{4.7}$$

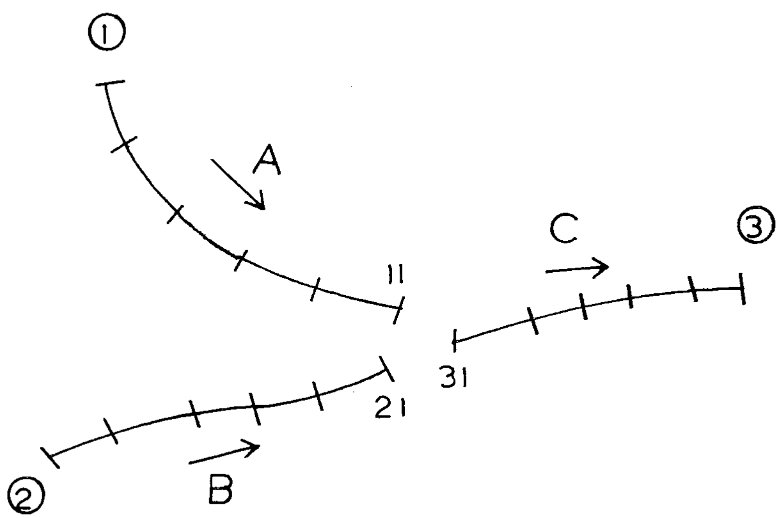

Fig. 2. First sweep directions in a branched channel system. ①, ②, ③ are external boundary points, 11, 21, 31 internal ones.

Now the sweep can continue from point 31 to ③, where again a boundary condition is applied to initiate the return sweep back to the junction and further on along the two branches.

If flow velocities are high so that the velocity head becomes significant, an energy level compatibility criterion should be used in the junction. Then a slightly more complicated expression is obtained on the right hand side of equations (4.6) and (4.7).

Internal boundary conditions has also to be specified when singular head losses in sudden channel expansions or the flow over a submerged weir are to be modelled. It is always important to pay enough attention to the selection of boundary points and boundary conditions, because in principle they affect the solution in all the computational points.

5. *Application to the Mänttä watercourse*

5.1 Description of the study area

A numerical model for the highly irregular Mänttä watercourse was required, because the regular flow from a hydropower plant in Mänttä changed the main flow direction in the Pieskansalmi sound every time the gates were opened, and thus it was polluted with water from a paper factory waste water outlet situated close to the hydropower plant. A map of the area and the used computational grid is shown in Fig. 3. The cross sectional areas, which ranged from 70 to 22 000 m^2, were obtained from somewhat inaccurate depth contour maps. It was not possible to remeasure the cross sections except for the two sounds close to the junction point 51. The main flow direction in the Kuorevesi branch is due to lateral inflow from several brooks and ditches discharging into this branch, and its magnitude will thus depend on the hydrological conditions.

5.2 Boundary conditions and estimations of coefficients

The points 11 and 21 have been treated as closed boundaries where the discharge is zero. At the point 41 (Mänttä) the flow varies according to the regulation and hence the discharge is given as a function of time. Discharge values were available at hourly intervals from the power plant. In the point 61 (Vilppula) there is a crest for which a rating curve exists and hence used as a boundary condition here. Points 31 and 51 are internal boundary points where water level compatibility is used.

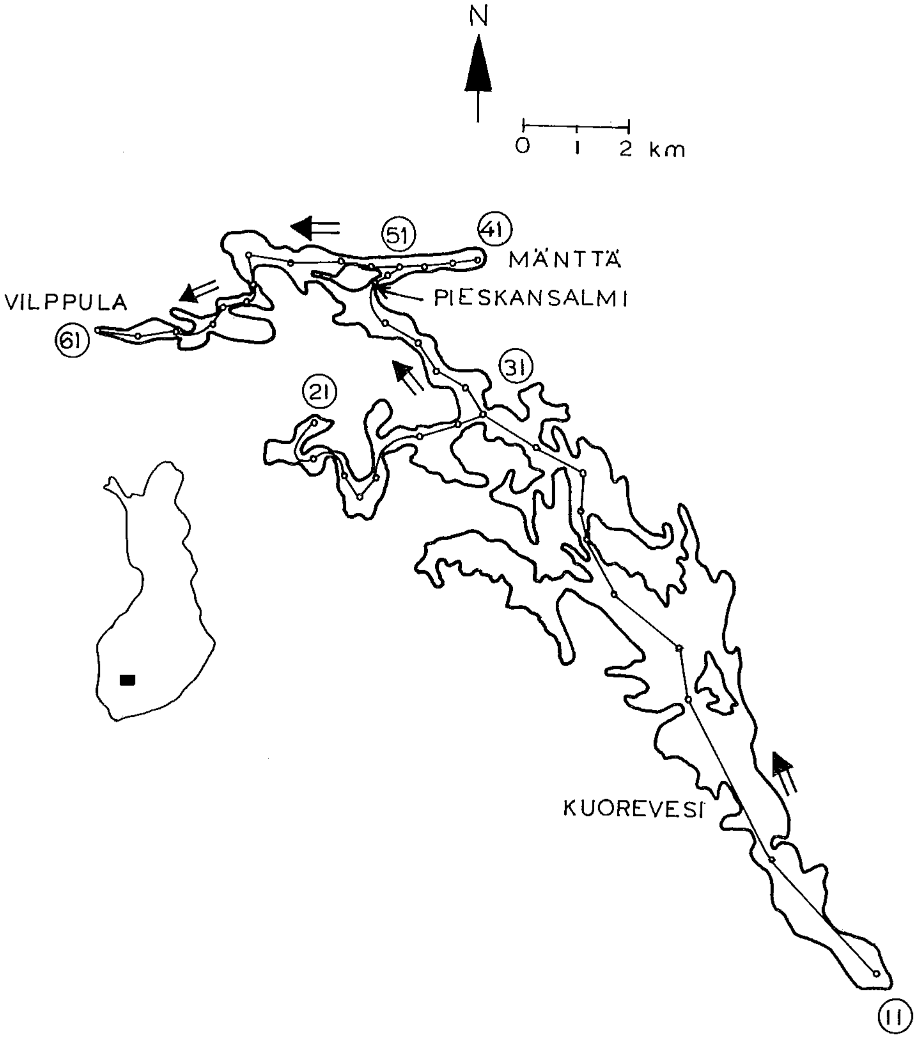

Fig. 3. The study area and computational points. ⇒ denotes natural flow direction, ○ boundary point.

Lateral inflow is significant only in the Kuorevesi branch where it is estimated from the mean discharge in Mänttä during the study period, the ration of discharge being equal to the ratio of the drainage areas of Kuorevesi and Mänttä. Hence the total discharge from the Kuorevesi branch was fixed to 1.6 m^3/s.

The conveyance in equation (2.2) was computed according to a Chezy resistance law. The value of the friction coefficient needed in the formula has been estimated using CHOW (1959), who gives a Manning roughness for wide rivers (surface width greater than 30 m during floods) between 0.025 and 0.06 corresponding to a Chezy coefficient of about 20...50 $m^{0.5}s^{-1}$. After a few test runs a *C*-value 33 was used throughout the model. The watercourse was ice covered during the study period, and one third of the surface width has been added to the wetted perimeter values of the cross sections, to account for the increased friction from the ice cover. This method is suggested in the handbook of RIL (1968). The value of the velocity distribution coefficient β was put to 1.1 in all cross sections.

6. *Measurements*

Current velocity measurements in the Pieskansalmi sound leading to Kuorevesi were done by the Hydrological Office of the National Board of Waters with a registering current meter (Aanderaa RCM4) at 10 min. intervals during 8.2.–27.2. 1979. Depth of measurement was 2,0 m, total water depth 4.1 m. The relation between measured current velocity and discharge in the Pieskansalmi sound was established through discharge measurements in the sound. The mean discharge of the Mänttä power plant was 8.1 m^3/s during this period, the variation being 1.4... 25.1 m^3/s. The watercourse was covered with ice of thickness 20...50 cm thus eliminating the influence of wind on the currents. The water level variations caused by the regulated discharge were known to be small, at most only a few mm, so the use of standard recording water gauges was considered inappropriate.

7. *Results and conclusions*

Mean hourly values of the predicted and observed discharges in the Pieskansalmi sound are shown in Fig. 4 for the last 10 days of the study period. It is seen that there is an immediate response to the regulation of the Mänttä power plant. The predicted peak discharges are somewhat too low, and the amplitude of the observed discharge variation is greater than the predicted. The correlation between the discharge of Mänttä and the observed discharge in the Pieskansalmi sound is 0.929 for the whole study period. This clearly indicates the dominating role of the regulation on discharge in this sound, and the limited effect of other dynamic considerations. The correlation between predicted and observed discharges for the same period is 0.970, which shows that the applied model is appropriate in this case. This is even more so, when one takes into account that the model was not calibrated in any particular way, by *e.g.* estimating the friction coefficient in different cross sections more accurately. Accurate measurements of some im-

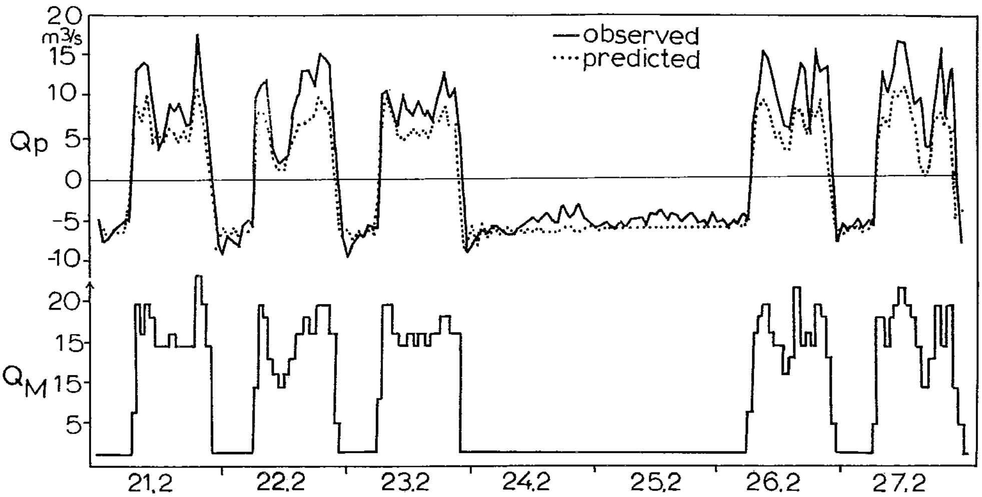

Fig. 4. Observed and predicted discharges Q_P in the Pieskansalmi sound. Positive values indicate flow towards Kuorevesi. Q_M is the regulated discharge of the Mänttä power plant.

portant cross sections should also improve the model. In any case it is encouraging that a fairly good agreement between measured and modelled discharges can be obtained in an irregular watercourse without too much computational effort.

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