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A THEORY OF VARIABLE WIND-DRIVEN CURRENTS WITH CONSTANT EDDY VISCOSITY

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Abstract

A solution of the Navier-Stokes equations has been derived assuming that the space available is horizontally unlimited and that the water depth is constant. As initial condition, it is assumed that the current velocity is given as a function of depth. As boundary condition, the wind stress is given as a function of time at the upper surface, and at the bottom the no-slip condition is valid. It is further assumed that the free surface has a constant inclination and that the vertical eddy viscosity is also constant. To check the theory, an optimation procedure was developed to find the best values for the eddy viscosity, the wind stress factor and the surface tilting including the direction of the tilting. Some cases were run through with variable success.

1. *Introduction*

Wind-driven currents have been theoretically dealt with in numerous papers. In Ekmans work (Ekman, [3]) a constant eddy viscosity is assumed. With an infinite depth and steady wind the surface current assumes an angle of $\pi/4$ to the right (on the northern hemisphere) of the wind direction, and downward from the surface the current velocity follows the well known hodograph, the Ekman spiral. Ekman formulated also the corresponding transient state problem, which was solved by Fredholm giving time dependent spirals which approached steady state when time went on. These steady state values for different depths form the customary Ekman spiral.

Subsequent scientists have mostly treated cases with steady wind and variable vertical eddy viscosity. FJELDSTAD [4] proposed that the eddy viscosity be proportional to the 3/4 power of the distance from the bottom while THOMAS [11] and WITTEN and THOMAS [12] assumed the eddy viscosity to be linearly dependent from the depth. Their hodographs were somewhat more realistic than the corresponding Ekman spirals. MURRAY [8] applied a constant and a power-formed eddy viscosity to near shore currents with some success. MADSEN [7], again, made a rather profound theoretical approach to the problem by assuming that the vertical eddy viscosity is proportional to the distance from the free surface. He found that in transient state the spiral formed was comprized into a much smaller angle than in the FREDHOLMS case and the deviation of the current direction from the wind direction in the drift current case was some 8°. Similar observations have been made in natural conditions.

In addition to the theoretical or partly theoretical works there are some works performed by computers, see *e.g.* FORISTALL [5] and SVENSSON [10]. Because this method is more flexible, it is easier to take into account more demanding conditions. Therefore it is understandable that the results too are more or less in accordance with natural conditions.

The purpose of this paper is to derive current velocities assumed to be horizontal as a function of depth and time-dependent wind when the vertical eddy viscosity is assumed to be constant. The current field is initially given as a function of depth. The upper surface may be inclined. To compare the theoretical and observed currents an optimation procedure was used.

2. *The problem*

The current field in a sea or lake obeys the Navier-Stokes equations which in complex form read as follows

$$\frac{\partial w}{\partial t} + \mathrm{i} f w = -\kappa g + \nu \frac{\partial^2 w}{\partial z^2} . \tag{2.1}$$

Here $w = w(t, z) = u + \mathrm{i}v$ is the complex velocity of the current assumed to be (nearly) horizontal, i is the imaginary unit, $f = 2\omega \sin\varphi$ is the Coriolis parameter, $\kappa = \kappa_x + \mathrm{i}\kappa_y$ is the constant complex tilting of the free surface and ν is a constant turbulent kinematic viscosity. Furthermore, the water density ρ is taken to be constant. Although not explicitly stated in the basic statements of the problem, the tilting of the surface may be caused by some object, a coast line or some other obstacle not too near to the regions considered. The initial and boundary conditions are

$$w(0,z) = w_0(z), \tag{2.2a}$$

$$w(t,-h) = 0, \tag{2.2b}$$

$$\frac{\partial w(t,0)}{\partial z} = \frac{\tau(t)}{\rho\nu}. \tag{2.2c}$$

The initial condition tells that at $t = 0$ the velocity is given as a function of z. The first boundary condition puts the current speed at the bottom = 0. The depth h is assumed to be constant. The second boundary condition gives a relationship between the vertical derivative of the current velocity at the surface $w(t,0)$, the surface stress $\tau = \tau(t)$, which is a given function of time, the density ρ and the kinematic viscosity ν.

3. *The mathematical solution of the problem*

The solution is obtained using Laplace transformation (CHURCHILL [2])

$$F = F(s,z) = \int_0^\infty \exp(-st)\cdot f(t,z)\,\mathrm{d}t = \mathcal{L}\{f(t,z)\}.$$

The transformed differential equation together with the transformations of the boundary conditions yield

$$\nu\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - (s+\mathrm{i}f)\,W = -w_0(z) + \frac{\kappa g}{s}, \tag{3.1a}$$

$$W(s,-h) = 0, \tag{3.1b}$$

$$\frac{\mathrm{d}W(s,0)}{\mathrm{d}z} = \mathcal{L}\left\{\frac{\tau(t)}{\rho\nu}\right\}, \tag{3.1c}$$

where

$$W = W(s,z) = \mathcal{L}\{w(t,z)\}. \tag{3.2}$$

General solution of the problem. To obtain the general solution of the problem, the general solution of the homogeneous equation will be added to the singular one of the inhomogeneous equation. The general solution of the linear homogeneous differential equation

$$\nu\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - (s+\mathrm{i}f)\,W = 0 \tag{3.3}$$

is

$$W_g(s, z) = \mathcal{L}\left\{\frac{\tau(\nu)}{\rho\nu}\right\}\mathcal{L}\{w_g^\star(t, z)\}, \tag{3.4a}$$

where the denotations

$$W_g^\star(s, z) = \mathcal{L}\{w_g^\star(t, z)\} = \frac{1}{r}\,\frac{\exp(r(z+h)) - \exp(-r(z+h))}{\exp(rh) + \exp(-rh)}, \tag{3.4b}$$

$$r^2 = (s + \mathrm{i}f)/\nu \tag{3.4c}$$

have been used. The general solution of the linear problem is thus

$$w_g(t, z) = \int_0^t \frac{\tau(t-t')}{\rho\nu}\, w_g^\star(t', z)\,\mathrm{d}t'. \tag{3.5}$$

Still the solution of (3.4b) has to be found. Using rules of operational calculus (CHURCHILL [2]), it can be shown that Eq. (3.4a) is equivalent to

$$\mathcal{L}\{(1/\nu)\exp(\mathrm{i}ft_1/\nu)\, w_g^\star(t_1/\nu, z)\} = f(s), \tag{3.6a}$$

$$f(s) = \frac{1}{q}\,\frac{\exp((z+h)q) - \exp(-(z+h)q)}{\exp(hq) + \exp(-hq)}, \tag{3.6b}$$

$$t_1 = \nu t, \tag{3.6c}$$

$$q^2 = s. \tag{3.6d}$$

Using the formula for the inverse Laplace transformation it ensues

$$(1/\nu)\exp(\mathrm{i}ft_1/\nu)\, w_g^\star(t_1, z) = (1/2\pi\mathrm{i})\lim_{\beta\to\infty}\int_{\alpha-\beta\mathrm{i}}^{\alpha+\beta\mathrm{i}} \exp(st_1) f(s)\,\mathrm{d}s, \tag{3.7}$$

where $\alpha > 0$ and the integrand in the right member is analytic in the half plane $\mathrm{R}(s) > \alpha$. The integration is thus performed in a complex plane.

Treatment of the transformation integral. Calculation of the integral in Eq. (3.7) takes place using the Cauchy residue theorem. The path of integration follows a closed curve Γ avoiding singular points of the integrand. The actual integration starts at the point A of the Fig. 3.1 and then follows the line segments Γ_1, Γ_2, Γ_3, Γ_4, Γ_5 back to A. The singularities of the integrand are poles situated at the zeroes of the divisor. Yet, at the origin, there is no singularity. The poles are thus situated at

$$s_n = -\pi^2(n + 1/2)^2/h^2. \quad (n = 0, 1, 2, \ldots) \tag{3.8}$$

The residues belonging to the poles are found by a limiting procedure.

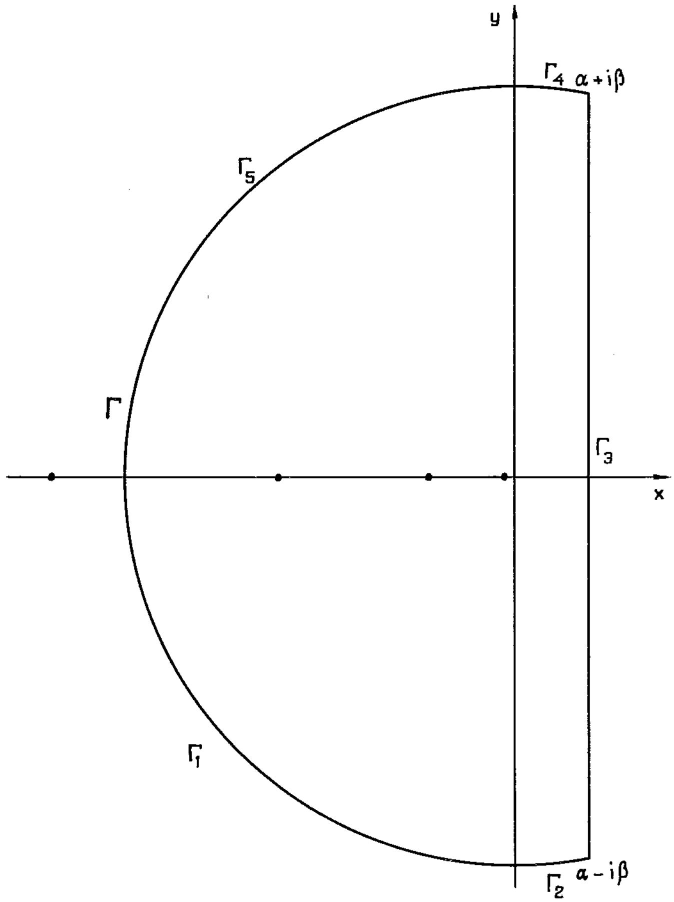


Fig. 3.1. Curve in the complex plane to be used to evaluate the integral in (3.7).

$$\rho(s_n) = \lim_{s \to s_n} (s - s_n) \exp(st_1) f(s) \tag{3.9a}$$

$$= (2/h) \exp(-\pi^2(n+1/2)^2 t_1/h^2) \cos((\pi z/h)(n + 1/2)). \tag{3.9b}$$

$$(n = 0, 1, 2, 3, \ldots)$$

It may be noted that there are no singularities in the right half plane $R(s) > 0$. According to the Cauchy residue theorem we then find

$$(1/2\pi i) \int_\Gamma \exp(st_1) f(s) \, ds = \sum_n \rho(s_n), \tag{3.10}$$

where the summation includes all residues of poles falling inside the curve Γ. The integral on the left hand side of (3.10) can be integrated piecewise. It is readily shown that integrals along the segments Γ_1, Γ_2, Γ_4, Γ_5, are of the form $O(|s|^{-1/2})$.

$$(1/2\pi i)\int_{\Gamma_j} \exp(st_1)f(s)\,ds = O(|s|^{-1/2}). \qquad (j = 1,2,4,5) \tag{3.11}$$

When the integrals are subtracted from (3.10), it is found that

$$\int_{\alpha-i\beta}^{\alpha+i\beta} \exp(st_1)f(s)\,ds = \sum_n \rho(s_n) + O(|s|^{-1/2}). \tag{3.12}$$

The general solution of the problem is then found by using Eqs. (3.5), (3.6c), (3.7), (3.9b), (3.12) to be

$$w_g(t,z) = \int_0^t 2\tau(t-t')/(\rho h) \sum_{n=0}^{\infty} \exp(-ift' - \pi^2 \nu t'(n+1/2)^2/h^2)\cos(\pi z(n+1/2)/h)\,dt'. \tag{3.13}$$

Singular solution of the problem. In the Eq. (3.1a) we have two inhomogeneous therms $w_0(z)$ and $\kappa g/s$. We treat them separately and try thus the solving of equations

$$\nu \frac{d^2 W_1}{dz^2} - (s+if)\,W_1 = -w_0(z), \tag{3.14a}$$

$$\nu \frac{d^2 W_2}{dz^2} - (s+if)\,W_2 = -\kappa g/s, \tag{3.14b}$$

where the subscripts refer to different solutions. To deal with the first equation, we assume that the initial velocity field be given as a Fourier series

$$w_0(z) = w(0,z) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi i n z/h). \tag{3.15}$$

It is thus

$$a_n = (1/h)\int_{-h}^{0} w_0(z)\exp(-2\pi i n z/h)\,dz. \tag{3.16}$$

It is further assumed that W_1 is the Laplace transform of w_1

$$W_1 = W_1(s,z) = \mathcal{L}\{w_1(t,z)\} \tag{3.17}$$

and that it can be expanded as Fourier series

$$w_1(t,z) = \sum_{n=-\infty}^{\infty} a_{1n} \exp(2\pi i n z/h) \tag{3.18}$$

with a Laplace transform

$$W_1 = \sum_{n=-\infty}^{\infty} A_{1n} \exp(2\pi i n z/h). \tag{3.19}$$

It is readily seen that Eq. (3.14) is satisfied by

$$A_{1n} = a_n/(s + if + 4\pi^2 n\nu/h^2). \tag{3.20}$$

Inverse Laplace transformation of Eq. (3.19) with (3.20) and (3.16) gives as the particular solution of the initial value problem

$$w_1(t,z) = (1/h) \sum_{n=-\infty}^{\infty} \exp(-(if + 4\pi^2 n^2 \nu/h^2)t) \int_{-h}^{0} w_0(z') \exp(2\pi i n(z-z')/h)\, dz'. \tag{3.21}$$

It is readily seen that Eq. (3.14b) has a solution

$$W_2 = -\kappa g(s(s + if))^{-1}. \tag{3.22}$$

The inverse Laplace transformation of (3.22) is

$$w_2(t,z) = -(\kappa g/f)\, i(\exp(-ift) - 1). \tag{3.23}$$

The solution of the whole problem is then obtained by adding the different solutions in (3.14), (3.21), (3.23)

$$w(t,z) = w_g(t,z) + w_1(t,z) + w_2(t,z). \tag{3.24}$$

4. *Data*

The data used in this work were furnished by Dr. M. LEPPÄRANTA at the Institute of Marine Research and were originally planned to settle some questions concerning ice and its movements, *c.f.* LEPPÄRANTA [6]. Occasionally, the data were judged to be suitable for the work at hand as well. They were collected by current meters lowered stepwise downwards through holes in certain ice floes to measure current velocities at different depths. The positions of the ice floes were determined by the navigational instruments of R/V Aranda, who was stationary anchored to the ice floe, where measurements were performed, Fig. 4.1. Lowering of the instrument was

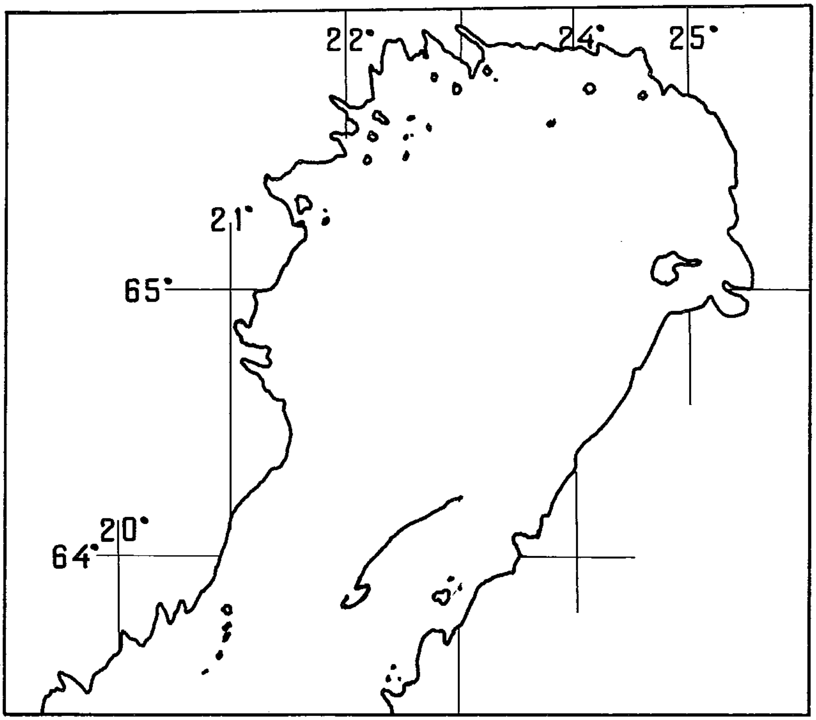


Fig. 4.1. Contours of the Bothnian Bay are shown. In the southern part of it there is a path of R/V Aranda showing her drift along with an ice floe to the southwest during one week in the first half of April, 1977.

carried at 1 m intervals for the 14 uppermost depths, then at 2 m intervals for depth up to 20 m and further down at 5 m intervals at times even as far down as 40 m. From these data and the floe velocities absolute current velocities were figured out. Wind data were also collected from meters in a mast installed nearby.

5. *Numerical considerations*

A computer programme was developed for the Burroughs 6700 computer of the University of Helsinki. In the first phase, the formula (3.24) above was used to calculate horizontal currents. As initial current field, measured values as a function of depth were given. Shear values at the upper surface were determined from two uppermost current measurements, because, the sea was covered by ice in regions where measurements were executed. From these values stress values were drawn by multiplying with the constant eddy viscosity. Tilting was taken as zero in this first phase. The results showed that the stress values used were far too large. This may be caused by very small viscosity values near the surface. As a remedy an additional factor k for the viscosity was introduced. The results showed also other strange features. It was thought that the reason could be the shore lines some distance apart. Another reason could be the ice ridges, especially the keels below the lower ice surface. As a remedy to this, it was thought, could be the tilting of the upper surface. Because the locations and forms of the keels were not specified by measurements, they could not be directly taken into account, although they certainly have some influence on currents. The equations contained in this way a number of undetermined quantities. Simple trial and error method seemed to be futile. Therefore it was decided to write a programme, which optimized the unsettled parameters, the eddy kinematic viscosity, the surface shear multiplier, the tilting angle of the surface and the direction of the tilting. The object then was to minimize the sum S below to find the best fit with the calculated current and the observed one.

$$S = \Sigma((u - u_0)^2 + (v - v_0)^2), \tag{5.1}$$

Here (u, v) means the velocity calculated and (u_0, v_0) the observed one at chosen depths. The data, as used to the problem at hand, had a defect, namely they covered at one and the same site a rather restricted time spell namely some few hours only. To recover this mishaps, some calculations were performed with extended time interval and the optimized data. Only the stress at the upper surface was determined in this case from the measured wind data using the well known square law (NEUMANN – PIERSON, [9]).

The numerical integration in (3.13) was performed by Gaussian quadrature method (ABRAMOVITZ – STEGUN, [1]). The reason, why Simpson rule was abandonned lies in the sum expression of Eq. (3.13), which does not converge at $t' = 0$.

6. *Results*

After the formula (3.13) was completed, a test for its correctness was carried out using constant wind blowing on a sea surface initially at rest. The result was a spiral of Fredholm type, EKMAN [3], Fig. 6.1. It was also possible to calculate cases with variable wind influencing on a sea surface initially at rest. In this numerical experiment the wind used is represented in Fig. 6.2 in a hodograph form. The results were similar to Figs. 6.3a, b, c. (The curves shown here have been calculated later on). In a) the surface current and in b) and c) the currents in the depths of 10.5 m and 21 m are plotted. The total depth was 42 m. Hereafter the programme was modified to

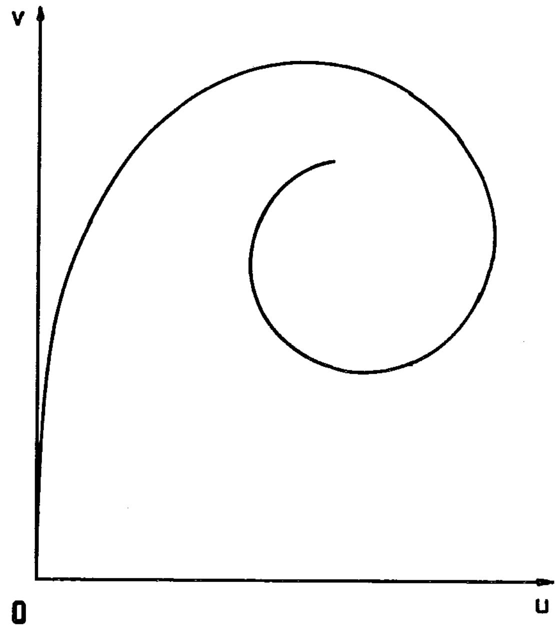


Fig. 6.1. The hodograph of the picture shows the development of surface current velocities determined according to the formula (3.13), when the current starts at rest and constant wind blows over the surface. It is assumed that the coasts and the bottom are so far apart that their influence on the velocities is not noticeable.

include the optimation procedure, the stress determination from shear values in the surface layers as an alternative to the wind stress calculation, the influence of the current values at start and that of the tilting. Results of different series of measurements are to be seen in Figs. 6.4a, b, c, d.

7. *Discussion*

In Fig. 6.3a we see the current start from zero and grow in the same direction as the wind blows at the time 0 h, Fig. 6.2. The wind speed grows then and its direction changes anticlockwise. The wind attains a first maximum at about 7...8 h. The current velocity grows slightly wiggling and its direction is at the maximum moment of the wind about 43° to the right from the wind direction. Then the wind subsides to a minimum at about 12 h and grows retaining its direction almost unchanged until 18 h. The current vector hodograph accomplishes here a loop and continues then to turn only slightly to the right. The maximum deviation of the current direction from the wind direction to the right is about 75°. All these phenomena are well in agree-

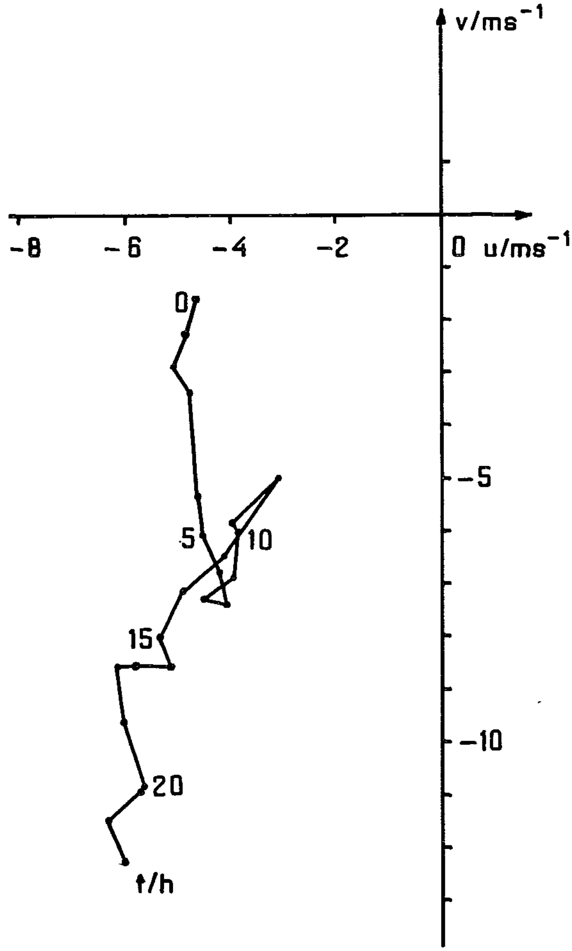
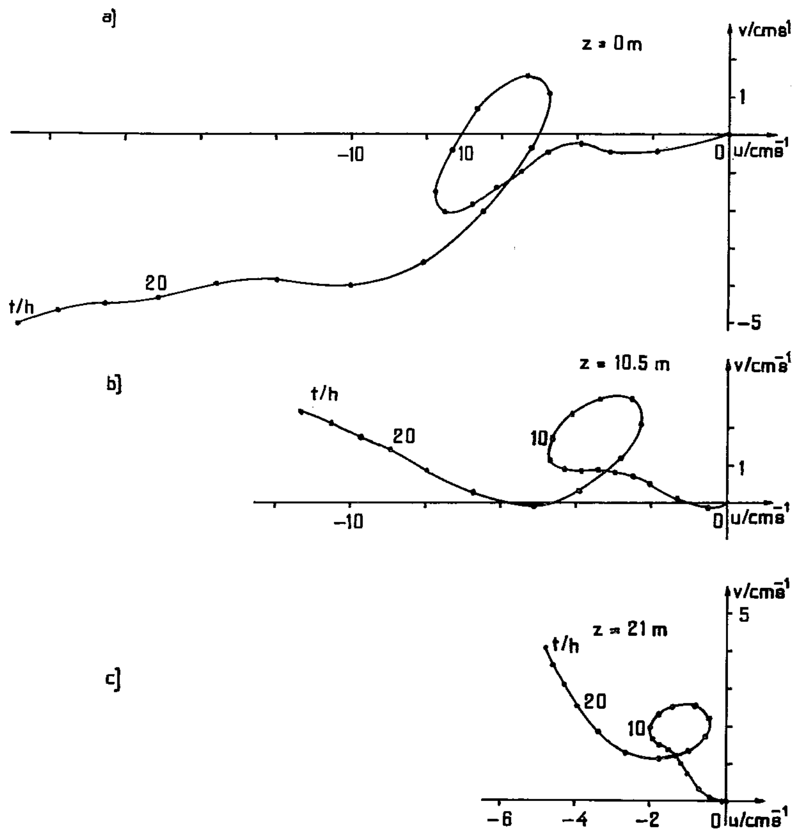


Fig. 6.2. Natural wind velocities (in a form of a hodograph) which were used in a numerical experiment to calculate the current shown in Figs. 6.3a, b, c.

ment with the constant viscosity assumption, EKMAN [3]. The subsurface currents are represented in Figs. 6.3b, c. The change of the general direction and reduction in speed are obvious. Also this phenomenon is in agreement with the simple theory of constant viscosity.

It seems thus obvious that the Eq. (3.13) can be used to compute currents in case of constant viscosity and constant depth when the wind is variable.

In Figs. 6.4a, b, c, d there are original observations shown as fulldrawn lines and optimated calculations plotted as dashed lines. The purpose of the optimation was



Figs. 6.3a, b, c. Calculation of the development of currents initially at rest in different depths according to the formula (3.13): a) at the surface, b) 10.5 m below the surface and c) 21 m below the surface, when the total depth is 42 m and the wind behaviour is that depicted in Fig. 6.2. The kinematic eddy viscosity was taken to be ν = 0.0194 m^2s^{-1}. The upper surface was assumed to be level.

to find some parameters to obtain the best fit between the observations and the calculations and at the same time check in what extend this can be done. The parameters optimized were the kinematic eddy viscosity, the factor by which the real surface stress was multiplied, the tilting of the sea surface and the direction of the horizontal component of the surface.

The optimation gave as the kinematic eddy viscosity in all the four cases astonishingly the same value 0.0194 m^2s^{-1} (with four decimal places). This agrees very well with the known value 0.02 m^2s^{-1} for the Gulf of Bothnia. The surface shear factor k had in the four cases (in the given order) the values 0.39, 0.20, 0.36, 0.18. The corresponding stresses calculated from the formula

$$\tau = k\rho\nu\partial|w|/\partial z$$

were 0.17, 0.31, 0.28, 0.43 $\mathrm{kgm^{-1}s^{-2}}$, whereas those calculated from windstress according to the formula

$$\tau = 0.002\ \mathrm{kgm^{-3}} \cdot w_a^2$$

gave 0.35, 1.29, 0.17, 0.30 $\mathrm{kgm^{-1}s^{-2}}$ respectively. The agreement is doubtful at least. It can be observed at the first glance that the real shears near the upper surface in the different cases are very different. Therefore it is understandable that the k-values too differ markedly. On the other hand, the sea areas in question were ice covered. Therefore the traction of wind on the sea surface is indirect. Indeed, it is known that the ice floes attain the speed due in a very short time, i.e. their sluggishness need not be taken into account in general, LEPPÄRANTA [6]. Besides, the factor in the stress formula may be slightly different from that used for the open sea.

Because the tilting in this case may not be of any especial interest, it is outlined here in broad lines only. The tilting may be compared with the expression

$$i = f|w|/g.$$

Using proper observed values, this formula gives $i = 1.8 \cdot 10^{-6}$, whereas the largest calculated value was $0.14 \cdot 10^{-6}$. This may mean that neighter the coast lines nor the keels of the ice ridges exert any appreciable influence on the flows measured.

The general shapes of the optimated velocities agree in broad outlines, but they are far from good agreement. One possible reason to this may be that factors other than the wind stress may have interfered, *e.g.* current pulses (turbulence, say) from other regions may have occasionally entered the region in question, at the irregular shapes of the current hodographs suggest. One circumstance that certainly has an influence on the results is the fact that the basic measurements are not simultaneous, but currents in each horizontal level have been measured separately. This means that the current determinations in any vertical section have been completed in some 20 to 30 minutes.

Because it is well known that the viscosity depend on surrounding factors, it is understandable that the constant viscosity model can not effectively cope with natural conditions.

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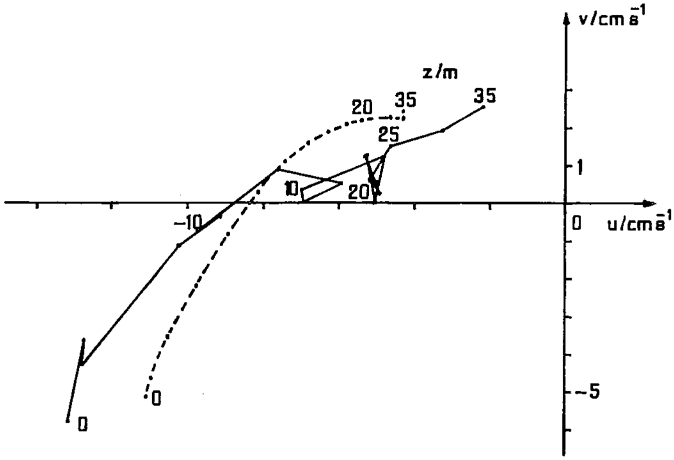
a)



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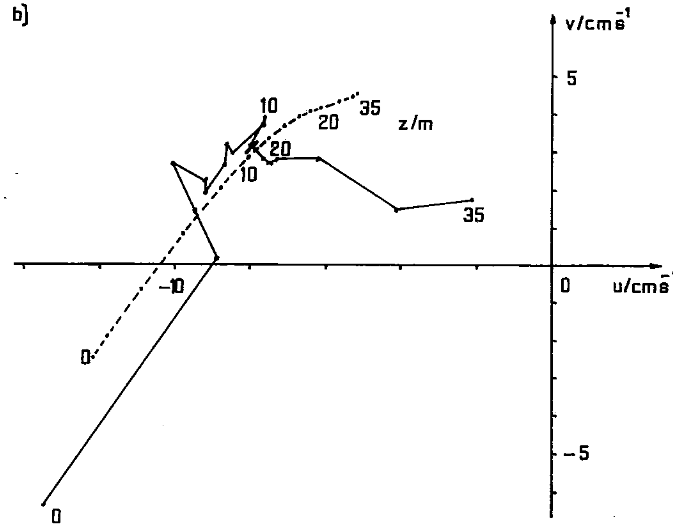
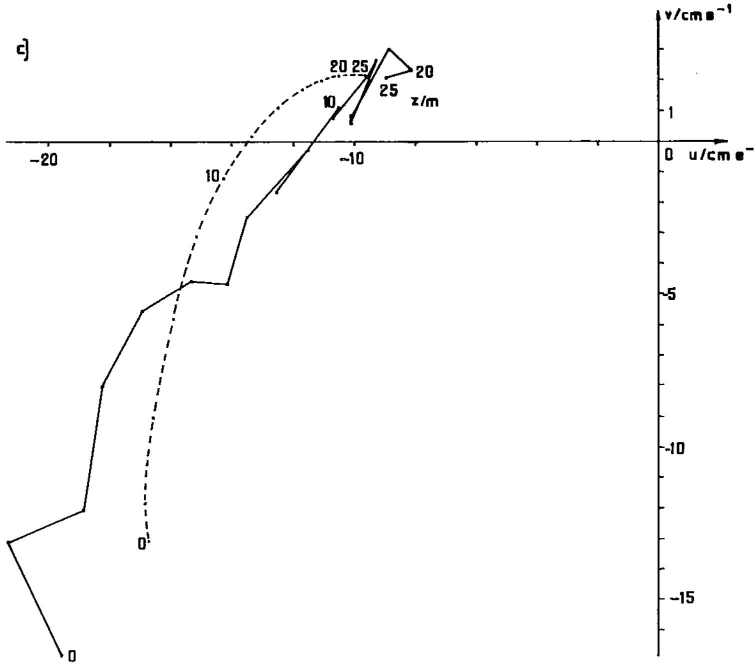


Figs. 6.4a, b, c, d. In these figures the original current hodographs are drawn in full lines and the hodographs determined using optimated values in dashed lines. The observations and the calculations correspond to a time 60 min after the initial values were introduced in the figure a). The corresponding times for the figures b), c) and d) are 100 min, 35 min and 125 min respectively.

c)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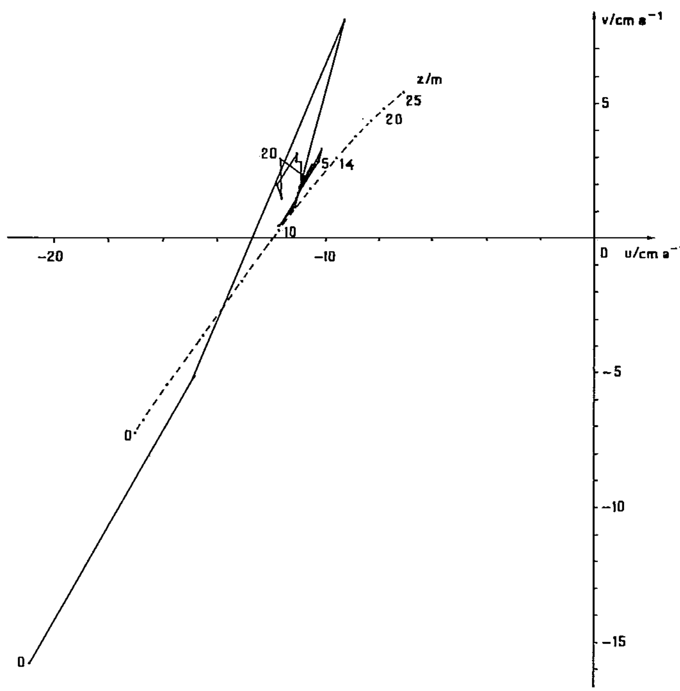
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the programme for the computer and performed the bulk of practical runnings. She has also drawn a number of curves during the present work. The author would also like to express his gratitude to all the many people who have participated in fruitful discussions.

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