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# ON THE CORRECTION TO BE APPLIED FOR VARIATIONS IN THE MUTUAL POSITION OF THE TRANSMITTER AND RECEIVER IN AIRBORNE ELECTROMAGNETIC MEASUREMENTS

by

E. J. Suoninen

Outokumpu Oy, Helsinki

Abstract

The effect of different types of changes in the mutual position of the transmitter and the receiver is investigated in two systems used in practice for airborne electromagnetic measurements: the single loop system and the rotary field system. The error in the measurement of the phase and amplitude of the signal is estimated, a certain statistical law being assumed to account for the most important variations. Some recent experiments on this problem with the single loop system are reported. The importance of certain constructional and aerodynamic factors is also discussed in both systems.

## 1. *Introduction*

Ever since the beginning of experiments on airborne electromagnetic measurements it has been recognized that one of the key problems is to control the mutual position of the transmitter and the receiver of the system [1]. In principle, two basic methods are available for correcting errors due to fluctuations in their relative position. The most natural method would be to keep the position constant with an accuracy corresponding to the accuracy desired of the measurement. If the transmitter and the receiver antennas are parallel, the field $H$ at the receiver is

$$H \sim \frac{1}{r^3} \tag{1}$$

where $r$ is the distance between the transmitter and the receiver. Logarithmic differentiation yields

$$\left|\frac{dH}{H}\right| = 3\left|\frac{dr}{r}\right| \tag{2}$$

The signal obtained is proportional to the field strength. Furthermore, it depends on the orientation of the receiver antenna with respect to the field, and hence also with respect to the transmitter antenna. The requirement of any specified level of accuracy in the airborne electromagnetic measurement can naturally be met by maintaining the mutual position of the transmitter and receiver constant with a corresponding accuracy. To achieve the accuracy necessary for successful prospecting has been considered a very difficult problem in the past. It is obvious that this is still true if there is no mechanical connection between the transmitter and the receiver, as in the case of two aircraft flying in tandem, the transmitter being in one plane, the receiver in the other.

## 2. *Experiments on stabilization of the mutual distance*

The optimal solution for the problem of stabilizing the relative position of the transmitter and the receiver has been studied in the experiments conducted recently by the Outokumpu Co. in cooperation with the Geological Survey of Finland. The measuring system consisted of a transmitter loop attached to a two-engine airplane and a receiver loop situated in a bird towed behind the plane. Both loops were in the same geometrical plane, which was tilted about 15 degrees with respect to the horizontal. The preliminary results show that with suitable construction of the cable and the bird and by flying at high enough speeds, it seems to be possible to keep the distance much more nearly constant than has hitherto been thought possible. There is hope for an accuracy of 1 per cent in the measurement of the amplitude at a distance of 200 meters, at least when flying in moderate weather conditions.

With respect to the phase of the signal received in the case of coplanar transmitter and receiver antennas, it can be concluded that errors introduced by moderate fluctuations in the mutual position are negligible as long as the assumption of a quasistationary field is valid, which is practically always the case in airborne electromagnetic measurements. As a result, the accuracy of the phase measurement is usually determined by other factors, mainly the electronics. The accuracy of about 0.1—0.2 per cent achieved in the equipment of the Geological Survey of Finland [1] is to be considered good.

One more point to be noted is that both the amplitude and the phase of the signal received in a system of two parallel antennas in the same plane are (in the absence of anomalies) only dependent on the length of the straight line between them, but not on its direction. This is a result of the cylindrical symmetry of the field of a single plane loop. Any transverse deviation of either the transmitter or the receiver is without effect on the measurement, as long as the distance remains constant. This condition is usually not fulfilled, if other geometries are used. Hence, a difference in the mutual orientation of the antennas as well as the inclusion of other field components in the measurement by the addition of antennas with differing orientations usually make the measurement sensitive to transverse movement of the antennas.

## 3. *Correction in the rotary field method*

Although it seems possible to improve the stabilization of the distance substantially from the present practice, the other alternative of handling the problem by a suitable correction procedure still remains an interesting possibility. The rotary field method ([2, 3, 4, 5, 6]) adds another, vertical antenna to the horizontal one both in the transmitting and receiving systems. It is well known how the amplitude and phase of the field are measured by summing the signals from the two receiving antennas through a 90 degree phase-shifting network and comparing the result with the primary field. The system compensates exactly for any variations in the distance as long as the condition stated above is fulfilled for both pairs of antennas, i.e. as long as the line intersecting the two perpendicular transmitter loop planes coincides with the corresponding line of the receiving system. However, if this condition is not fulfilled, an error is introduced in the measurement of either the amplitude or the phase or both, even if the distance has not been changed. The magnitude of this error can be evaluated in the following way:

Let us first consider the symmetrical case (Fig. 1). We assume in the following that there are no anomalies present. Let us denote the voltage induced in the vertical and the horizontal receiver coils with $e$ and $e'$,

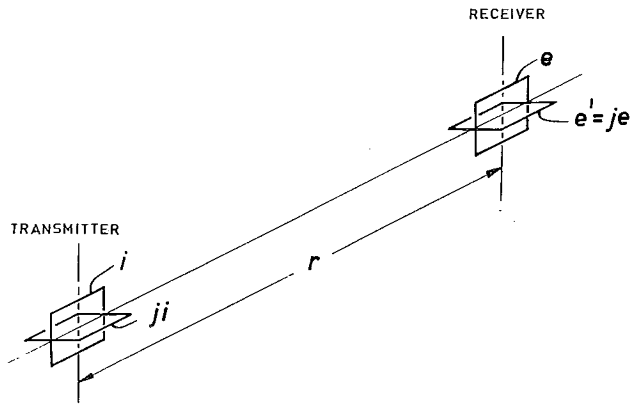


Fig. 1. Rotary field method. Symmetrical case.

respectively. They have the same amplitude, but the phases differ by 90 degrees:

$$e' = je \tag{3}$$

The next step in the rotary field method is to form the quantity $e + je'$. In the symmetrical case we have

$$e + je' = e + j(je) = e - e = 0\,. \tag{4}$$

In the nonsymmetrical case of Fig. 2, the two voltages have been changed. The components of the voltages induced in the receiver coils are as follows:

$$\begin{aligned} e_{vv} &= e\left(1 - \frac{3x^2}{r^2}\right) \qquad & e_{hv} &= e'\,\frac{3xy}{r^2} \\ e_{vh} &= e\,\frac{3xy}{r^2} & e_{hh} &= e'\left(1 - \frac{3y^2}{r^2}\right) \end{aligned} \tag{5}$$

The first subscript refers to the transmitting antenna which radiates the field component in question and the second subscript refers to the receiver antenna in which the voltage component is induced (e.g. $e_{vh}$ = voltage induced in the horizontal receiving antenna due to the field radiated by the vertical transmitting antenna). The quantity corresponding to $e + je'$ is now

$$\begin{aligned} (e_{vv} + e_{hv}) + j(e_{vh} + e_{hh}) &= \left[e\left(1 - \frac{3x^2}{r^2}\right) + je\,\frac{3xy}{r^2}\right] \\ + j\left[je\left(1 - \frac{3y^2}{r^2}\right) + e\,\frac{3xy}{r^2}\right] &= e\left[\frac{3(y^2 - x^2)}{r^2} + j\,\frac{6xy}{r^2}\right] \end{aligned} \tag{6}$$

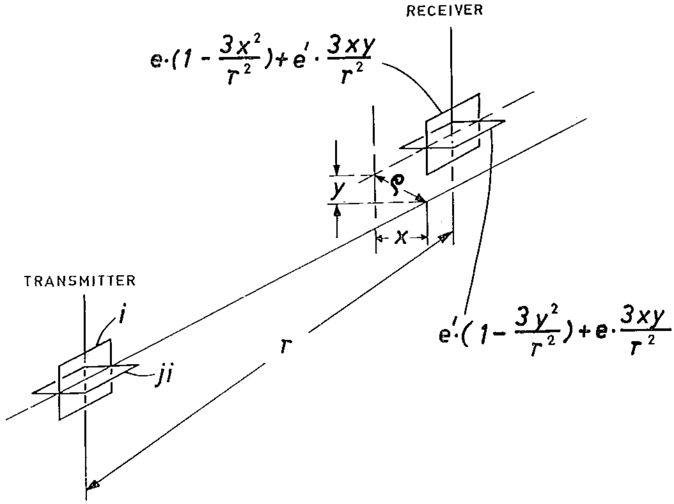


Fig. 2. Rotary field method. Nonsymmetrical case.

The errors introduced in the measurement of the amplitude and phase are therefore

$$\Delta e_{re.} = e\,\frac{3(y^2 - x^2)}{r^2} \qquad \Delta e_{im.} = e\,\frac{6xy}{r^2} \tag{7}$$

We see that if $|x| = |y|$, $\Delta e_{re.} = 0$. On the other hand, if either $x$ or $y$ is $= 0$, $\Delta e_{im.} = 0$. A movement in the $xy$-plane away from the correct position along lines of 45° slope with the horizontal does not cause any amplitude error, whereas a movement in the horizontal or vertical direction is free from any phase error.

In practice, the receiver antenna assembly moves in an irregular way around the axis of alignment. The nature of this movement is dependent on the weather conditions, aerodynamic properties of the plane or bird carrying the receiver, etc. Where the receiver is in an aeroplane, the movement is naturally also affected by the corrective measures taken by the pilot. In general, it is obvious that the deviations are the more frequent, the smaller their amplitude in the horizontal and vertical directions. Hence, it seems plausible to assume that the probability of finding the assembly at any given time at the position $(x, y)$ can be represented by the two-dimensional Gaussian distribution function

$$P(x\,,\,y) = e^{-(ax^2 + by^2)} \tag{8}$$

The constans $a$ and $b$ can be defined in terms of the standard deviations $x_m$ and $y_m$ representing a probability $\frac{1}{e}$:

$$x_m = \frac{1}{\sqrt{a}}$$

$$y_m = \frac{1}{\sqrt{b}}$$

$$P(x_m\,,\,0) = P(0\,,\,y_m) = \frac{1}{e}$$

The root mean square value of the error signal can be calculated by computing the weighted root mean square values of the error signals at different points $(x, y)$. The weighting function to be used is the distribution function $P(x, y)$. Since the functions (7) and (8) are all even, the area of integration can be limited to the first quadrant of the $xy$-plane. The evaluation of the root mean square values of the error signals associated with the measurement of the amplitude and the phase is carried out in Appendix I. The following results are obtained:

$$\frac{\overline{\Delta e_{re.}}}{e} = \frac{3}{2r^2} \cdot \frac{\sqrt{3a^2 - 2ab + 3b^2}}{ab} = \frac{3}{r^2}\sqrt{\frac{3}{4}(x_m^4 + y_m^4) - \frac{1}{2}x_m^2 y_m^2}$$

$$\frac{\overline{\Delta e_{im.}}}{e} = \frac{3}{r^2\sqrt{ab}} = \frac{3}{r^2}\,x_m y_m \qquad (9)$$

The rotary field method is also affected by a rotation of the transmitter assembly around the vertical axis, because such a movement violates the condition of coinciding intersecting lines. However, the errors due to rotation are probably of minor importance compared with the errors treated above. The single loop system is, of course, insensitive to this type of error owing to the absence of vertical antennas.

## 4. *Correction in the single loop method*

In the case of one transmitter and one receiver antenna which are in the same plane, it is seen once again from equations (5) that a deviation along the $x$-axis has no effect on the signal ($e_{hh}$ does not depend on $x$). In this case, however, there are the two other corrections to be considered, which are automatically made in the rotary field method. The effect of variations $z$ in the direction of the alignment axis ($z$-axis) can be represented by expressing the field strength as a Maclaurin series at $z = 0$. On the basis of (1), we get

$$H(z) = H(0)\left[1 - 3\frac{z}{r} + 12\frac{z^2}{r^2} - \dots\right] \tag{10}$$

The other effect is the effect of a rotation of the receiving loop around the axis of alignment through an angle $\varphi$. This reduces the projection of the antenna, and hence the signal received, from a relative value of 1 to

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \dots \tag{11}$$

A simultaneous displacement with respect to $y, z$ and $\varphi$ results in an error in the amplitude measurement

$$\Delta e_{re.} = e - e\left[\left(1 - \frac{3y^2}{r^2}\right)\left(1 - 3\frac{z}{r} + \dots\right)\left(1 - \frac{\varphi^2}{2} + \dots\right)\right] \tag{12}$$

By carrying out the multiplication in the square brackets we see that $y$ and $\varphi$ appear only in terms of second order or higher, whereas $z$ appears in the first order term $-3\frac{z}{r}$. If we limit ourselves to the first order approximation, $y$ and $\varphi$ disappear and (12) is simplified to

$$\Delta e_{re.} \approx e \cdot \frac{3z}{r} \tag{13}$$

Hence, the deviation along the axis of alignment dominates the amplitude error in this case. If, analogically with (8), we define a function

$$P(z) = e^{-cz^2} \tag{14}$$

to represent the probability of deviation along the $z$-axis, the root mean square error voltage can be calculated. This is done in Appendix II. The result obtained is

$$\frac{\overline{\Delta e_{re.}}}{e} = \frac{3}{\sqrt{2r}\sqrt{c}} = \frac{3}{\sqrt{2r}} \cdot z_m\,, \tag{15}$$

where $z_m = \frac{1}{\sqrt{c}}$.

## 5. *Discussion*

The assumption of a Gaussian distribution of the deviations probably gives a qualitively correct picture of the variations if the transmitter and the receiver are in two planes without mechanical connection with each other. However, if the receiver is being towed in a bird, the constraining effect of the cable possibly tends to make the distributions tail off faster, at great deviations, than is predicted by the Gaussian distribution. In order to demonstrate the resulting effect on the error voltage, the error voltage was re-evaluated for the rotary field method using two other distribution functions which presumably correspond better to the situation in the case of a towed bird. The results are given in Appendix III. They show that cutting off the tail of the distributions greatly reduces the error voltage. This strongly suggests the usefulness of a mechanical connection between the transmitter and the receiver even in the rotary field method. That a similar effect would be exerted in the single loop method is obvious.

The error signal appears as irregular noise over a certain frequency spectrum. The spectral distribution is largely determined by the acceleration of the systems carrying the antennas. Apart from vertical movements due to sudden variations in air pressure, the movements are mostly relatively slow, according to observations made during the experiments described in Section 2. The bulk of the noise is therefore concentrated within a frequency range of a few tenths of cps.

Figs. 3 and 4 give the numerical values of the functions (9) and (15) calculated for a typical distance $r = 250$ m. The average deviations have been taken as variables instead of $a$, $b$ and $c$ to facilitate the application of the graphs to practice.

The results demonstrate the good stability of the amplitude measurement in the rotary field method as compared with the method of single loops. As a typical example of the circumstances prevailing in practice during good weather, the following numerical values can be considered:

Rotary field method: $r = 250$ m

$x_m = 10$ m

$y_m = 5$ m

$$\frac{\overline{\Delta e_{re.}}}{e} \approx 0.39 \text{ per cent}$$

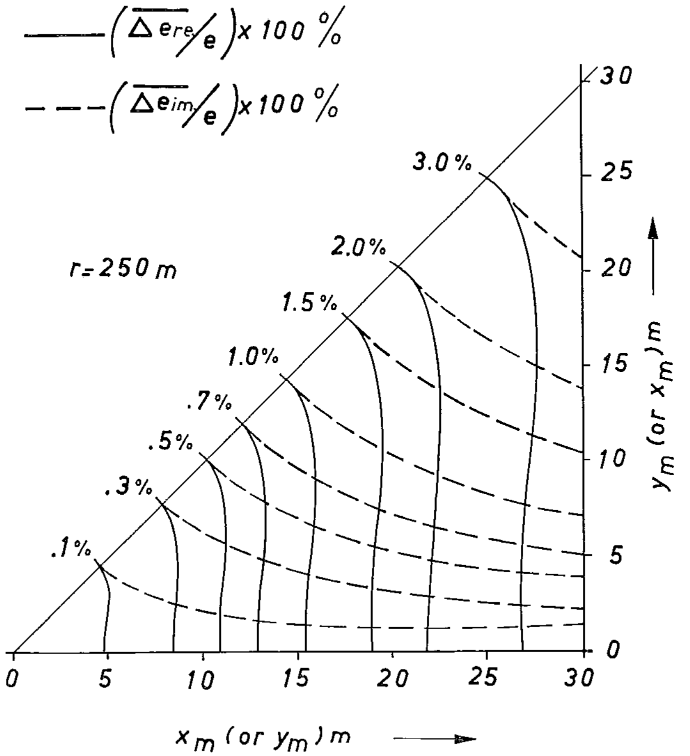


Fig. 3. Errors in the rotary field measurement of the amplitude and the phase as functions of the average deviations. A Gaussian distribution of the deviations has been assumed. $r$ = distance between the transmitter and the receiver.

Single loop method: $r = 250$ m
(receiver in bird) $z_m = 2$ m

$$\frac{\overline{\Delta e_{re.}}}{e} \approx 1.7 \text{ per cent.}$$

With respect to the accuracy of the phase measurement, the situation is reversed. With the above geometry, the rotary field method gives the accuracy

$$\frac{\overline{\Delta e_{im.}}}{e} \approx 0.24 \text{ per cent,}$$

which is already of at least the same order or even slightly worse than the value typical of the electronic noise with the single loop method (Section 2).

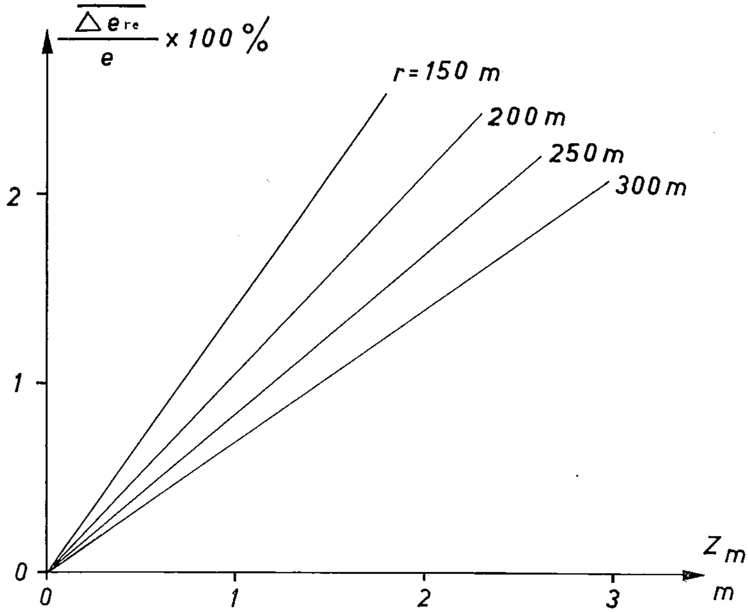


Fig. 4. Error in the single loop measurement of the amplitude as a function of the average deviation of distance. A Gaussian distribution of the deviation has been assumed. $r$ = distance between the transmitter and the receiver.

In moderate winds or locally varying pressure conditions, the above values become somewhat worse with both methods. Let us consider the rotary field method in a case in which the deviations are double their previous values, whereas the distance remains the same. Owing to the quadratic nature of the errors, their numerical values are quadrupled:

$$r = 250 \text{ m}$$
$$x_m = 20 \text{ m}$$
$$y_m = 10 \text{ m}$$

$$\frac{\overline{\Delta e_{re.}}}{e} \approx 1.6 \text{ per cent} \qquad \frac{\overline{\Delta e_{im.}}}{e} \approx 0.96 \text{ per cent}$$

Assuming that the average deviation in the $z$-direction is also doubled, we get for the amplitude error in the single loop system correspondingly

$$r = 250 \text{ m}$$
$$z_m = 4 \text{ m}$$

$$\frac{\overline{\Delta e_{re.}}}{e} \approx 3.4 \text{ per cent.}$$

This is a prohibitively large value and typical of a case with considerable slack motion. It must be remembered, however, that the tighter the cable, the more compressed is the distribution of the deviations in the direction of increasing distance. The mathematical evaluation of the resulting effect is difficult, owing to the different nature of the deviation distribution in the directions of increasing and decreasing distance. It can be concluded from the results of Appendix III, however, that a strong decrease from the value 3.4 per cent is to be expected.

The treatment of the amplitude error in the single loop system was based on the assumption that equation (13) is valid, i.e. that the error due to the deviations in the $z$-direction is much greater than the other error terms in the complete expression (12). As explained above, this term is decreased by tightening the cable. Even if the effect of the $z$-deviations is greatly reduced by a sufficiently tight cable, the effect of the $y$- and $\varphi$-deviations remains. Of these, the $y$-deviation is probably the more important one. Under such circumstances, the accuracy of the measurement becomes practically identical with the accuracy of the rotary field system. The single loop system cannot, of course, ever be made any better than the rotary field method.

As a result of all the above considerations, we arrive at the perhaps surprising result that the better the weather conditions, the more favorable is the rotary field method in comparison with a single loop system with bird. It is assumed, however, that the cable can be made so tight that its spring action is largely eliminated. It this is the case, the final choice between the two systems depends on whether the main emphasis is to be placed on the accuracy of the amplitude or of the phase measurement.

An additional aspect to be considered in the choice of method is the constructional and aerodynamic properties of the different equipments. In this respect the single loop system is obviously superior to the rotary field system. Difficulties are often experienced in attaching a large vertical transmitter loop to the airplane. The reason is that the vertical dimensions of airplanes are relatively small. As a result in the rotary field method one practically always has to sacrifice transmitting efficiency because of reducing the size of the vertical and consequently also of the horizontal loop.

## 6. *Summary*

The effects of changes in the mutual position of the transmitter and receiver are discussed on an experimental and theoretical basis. In the rotary field method, there is always an error signal background in both the amplitude and phase measurements due to the statistical translational movement of the receiver antennas in the plane perpendicular to the direction of flight. A cable connection between the transmitter and the receiver probably tends to reduce the error voltage because it limits the maximal sidewise movements. In the single loop system, the phase measurement is practically unaffected by any changes in the relative position, whereas the amplitude measurement is mainly disturbed by changes in the distance between the receiver and the transmitter. In checking the effect of the error, high tension in the towing cable and subsequent elimination of the slackening action are essential.

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Fig. 3 was drawn on the basis of numerical computations made by Mr K. FALLENIUS on the Elliott 803 computer of SUOMEN KAAPELITEHDAS OY (Finnish Cable Works Ltd.). The author wishes to express his gratitude to all persons involved.

## REFERENCES

1. PURANEN, M., 1959: Aerosähköisistä mittausmenetelmistä. *Thesis for the Lic. Phil. Degree.* University of Helsinki.
2. TORNQVIST, B. H., 1956: Ueber elektromagnetische Prospektion vom Flugzeug aus. *Freiberger Forschungshefte. C.* 29. 19—25.
3. TÖRNQVIST, G., 1958: Some practical results of airborne electromagnetic prospecting in Sweden. *Geophysical Prospecting.* 6. 112—126.
4. HEDSTRÖM, E. H. and D. S. PARASNIS, 1958: Some model experiments relating to electromagnetic prospecting with special reference to airborne work. *Ibid.* 6. 322—341.
5. PATERSON, N. R., 1959: Comments on paper entitled »Some Model Experiments . . .» by E. H. Hedström and D. S. Parasnis. *Ibid.* 7. 435—447.
6. HEDSTRÖM, E. H. and D. S. PARASNIS, 1959: Reply to comments by N. R. Paterson. *Ibid.* 7. 448—470.
7. 1957: HANDBOOK OF CHEMISTRY AND PHYSICS. 39th Ed. Chemical Rubber Publishing Co.

*Appendix I. Calculation of the Root Mean Square Errors in the Case of the Rotary Field Method*

a) $$\overline{\Delta e_{re.}} = \sqrt{\frac{\int\limits_0^\infty\int\limits_0^\infty [\Delta e_{re.}(x\,,\,y)]^2\, P(x\,,\,y)dxdy}{\int\limits_0^\infty\int\limits_0^\infty P(x\,,\,y)dxdy}} =$$

$$\sqrt{\frac{\int\limits_0^\infty\int\limits_0^\infty \left[\frac{3(y^2-x^2)e}{r^2}\right]^2 e^{-(ax^2+by^2)}\,dxdy}{\int\limits_0^\infty\int\limits_0^\infty e^{-(ax^2+by^2)}\,dxdy}} =$$

$$\frac{3e}{r^2}\sqrt{\frac{\int\limits_0^\infty\int\limits_0^\infty (y^2-x^2)^2 e^{-(ax^2+by^2)}\,dxdy}{\int\limits_0^\infty\int\limits_0^\infty e^{-(ax^2+by^2)}\,dxdy}} = \frac{3e}{r^2}\sqrt{\frac{A}{C}}$$

The double integrals $A$ and $C$ can be changed into expressions containing only simple integrals by developing and separating the integrands:

$$A = \int\limits_0^\infty\int\limits_0^\infty (y^4 - 2y^2x^2 + x^4)\,e^{-(ax^2+by^2)}\,dxdy =$$

$$\int\limits_0^\infty\int\limits_0^\infty y^4 e^{-(ax^2+by^2)}\,dxdy - 2\int\limits_0^\infty\int\limits_0^\infty y^2x^2 e^{-(ax^2+by^2)}\,dxdy +$$

$$+ \int\limits_0^\infty\int\limits_0^\infty x^4 e^{-(ax^2+by^2)}\,dxdy =$$

$$= \left[\int\limits_0^\infty y^4 e^{-by^2}\,dy\right]\left[\int\limits_0^\infty e^{-ax^2}dx\right] - 2\left[\int\limits_0^\infty y^2 e^{-by^2}\,dy\right]\left[\int\limits_0^\infty x^2 e^{-ax^2}\,dx\right]$$

$$+ \left[\int\limits_0^\infty x^4 e^{-ax^2}\,dx\right]\left[\int\limits_0^\infty e^{-by^2}\,dy\right]$$

The values of the integrals in the square brackets can be obtained from the following formulas (see f.ex. [7], p. 275):

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^{2n}\,e^{-ax^2}\,dx = \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^{n+1}\,a^n}\sqrt{\frac{\pi}{a}}$$

By inserting these in the expression for $A$, we obtain

$$A = \frac{\pi}{16(ab)^{5/2}}\,[3a^2 - 2ab + 3b^2]$$

In the same way, we obtain

$$C = \left[\int_0^\infty e^{-ax^2}\,dx\right]\left[\int_0^\infty e^{-by^2}\,dy\right] = \frac{\pi}{4\sqrt{ab}}$$

and finally

$$\frac{\overline{\Delta e_{re.}}}{e} = \frac{3}{r^2}\sqrt{\frac{A}{C}} = \frac{3}{2abr^2}\sqrt{3a^2 - 2ab + 3b^2}$$

b)

$$\overline{\Delta e_{im.}} = \sqrt{\frac{\int_0^\infty\int_0^\infty [\Delta e_{im.}(x\,,\,y)]^2\,P(x\,,\,y)\,dx\,dy}{\int_0^\infty\int_0^\infty P(x\,,\,y)\,dx\,dy}} =$$

$$e\sqrt{\frac{\int_0^\infty\int_0^\infty \left[\frac{6xy}{r^2}\right]^2 e^{-(ax^2+by^2)}\,dx\,dy}{\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\,dx\,dy}} = \frac{6e}{r^2}\sqrt{\frac{B}{C}}$$

with

$$B = \int_0^\infty\int_0^\infty x^2y^2e^{-(ax^2+by^2)}\,dx\,dy = \left[\int_0^\infty x^2e^{-ax^2}dx\right]\left[\int_0^\infty y^2e^{-by^2}\,dy\right] = \frac{\pi}{16\,(ab)^{3/2}}$$

$C$ is the same as in a). Hence, we obtain

$$\frac{\overline{\Delta e_{im.}}}{e} = \frac{6e}{r^2}\sqrt{\frac{B}{C}} = \frac{3}{r^2\sqrt{ab}}$$

*Appendix II. Calculation of the Root Mean Square Errors in the Case of Single Coplanar Loops*

We obtain, in the same way as in Appendix I,

$$\Delta e_{re.} = e\sqrt{\frac{\int\limits_0^\infty \left(\frac{3z}{r}\right)^2 e^{-cz^2}\,dz}{\int\limits_0^\infty e^{-cz^2}\,dz}} = \frac{3e}{r}\sqrt{\frac{D}{E}}$$

where
$$D = \int\limits_0^\infty z^2 e^{-cz^2}\,dz = \frac{1}{4c}\sqrt{\frac{\pi}{c}}$$

$$E = \int\limits_0^\infty e^{-cz^2}\,dz = \frac{1}{2}\sqrt{\frac{\pi}{c}}$$

Hence
$$\frac{\overline{\Delta e_{re.}}}{e} = \frac{3}{\sqrt{2}\cdot r\cdot\sqrt{c}}$$

*Appendix III. Effect of Varying the Distribution Function*

Figure 5 shows graphically three different distribution functions:

$$e^{-ax^2} = 1 - ax^2 + \frac{a^2x^4}{2} - \ldots\ldots$$

$$\cos(\sqrt{2a}\,x) = 1 - ax^2 + \frac{a^2x^4}{6} - \ldots\ldots \qquad \left(x \leq \frac{\pi}{2\sqrt{2a}}\right)$$

$$f(x) = 1 - ax^2 \qquad \left(x \leq \frac{1}{\sqrt{a}}\right)$$

The first two terms on the right hand sides of the equations are the same for all, whereas the contribution of the higher order terms is reduced on going down the list. This is reflected by the similarity of the first parts of all the curves in Fig. 5 and the correspondingly increasing rate of decay with increasing values of $x$. Hence, a reduction of the higher terms of the distribution function is a qualitative representation of the effect of a constraint such as that described in Section 5.

To illustrate the effect of a constraint, the formulas for $\frac{\overline{\Delta e_{re.}}}{e}$ were also calculated for the two other distribution functions in the case of the rotarry field in the same way as was done in Appendix I for the Gaussian distribution. The comparison of the results in terms of the same $x_m$ and $y_m$ can be given in the form of two coefficients $K_1$ and $K_2$:

$$\frac{\overline{\Delta e_{re.}}}{e} = \frac{3}{r^2}\sqrt{K_1(x_m^4 + y_m^4) - K_2 x_m^2 y_m^2}$$

| Distr. function | $K_1$ | $K_2$ |
|---|---|---|
| $e^{-ax^2}$ | $\frac{3}{4} = 0{,}75$ | $\frac{1}{2} = 0{,}5$ |
| $\cos(\sqrt{2a}\cdot x)$ | $\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24 = 0{,}0925$ | $\left[\left(\frac{\pi}{2}\right)^2 - 2\right]^2 = 0{,}218$ |
| $1 - ax^2$ | $\frac{3}{35} = 0{,}0857$ | $\frac{2}{25} = 0{,}08$ |

It is seen that both $K_1$ and $K_2$ are strongly reduced on going over to steeper distributions. This results in a great decrease in the value of the error signal which is demonstrated by the following numerical example:

$$\begin{cases} x_m = 10 \text{ m} \\ y_m = 5 \text{ m} \end{cases}$$

| Distr. function | $\frac{\overline{\Delta e_{re.}}}{e} \cdot 100$ per cent |
|---|---|
| $e^{-ax^2}$ | 0,39 |
| $\cos(\sqrt{2a}\,x)$ | 0,10 |
| $1 - ax^2$ | 0,13 |

It can be concluded that in the case of a Gaussian distribution the tails of the distribution, i.e. the occasional large deviations, are responsible for most of the amplitude error.

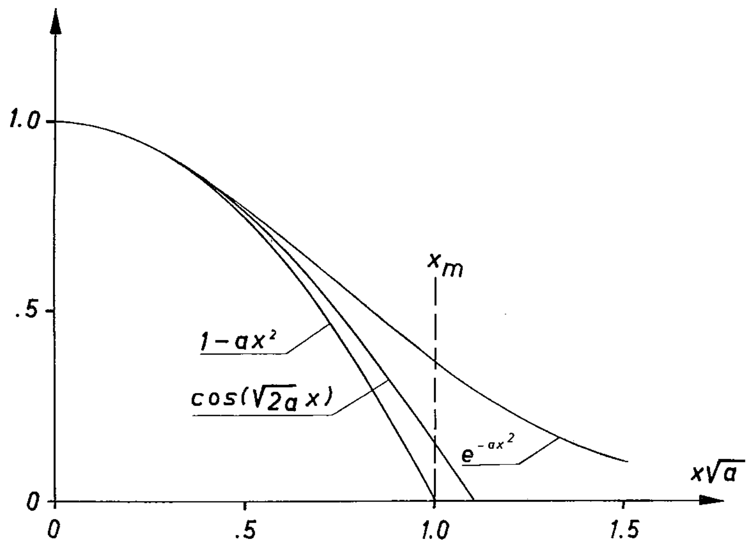


Fig. 5. Graphical representation of the different choices for the distribution function.